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Ward identities and renormalization of general gauge theories

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Abstract
We introduce the concept of general gauge theory which includes Yang–Mills models. We use the framework of the causal approach and show that the anomalies can appear only in the vacuum sector of the identities obtained from the gauge invariance condition by applying derivatives with respect to the basic fields. For the Yang–Mills model we provide these identities in the lowest orders of the perturbation theory and prove that they are valid. The investigation of higher orders of the perturbation theory is still an open problem.

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1. Introduction

The causal approach to perturbative renormalization theory by Epstein and Glaser [23, 24] significantly simplifies the conceptual and computational aspects for quantum electrodynamics [13, 28, 42], Yang–Mills theories [1, 3, 9, 10, 16, 17, 19–21, 25–27, 30, 35, 37, 43, 44], gravity [32, 33, 47], the analysis of scale invariance [29, 40], Wess–Zumino model [31], etc. In this approach one uses exclusively the Bogoliubov axioms of renormalization theory [8] imposed on the scattering matrix: this is an operator acting in the Hilbert space of the model, which is a Fock space generated from the vacuum by the quantum fields corresponding to the particles of the model. If one considers the S-matrix as a perturbative expansion in the coupling constant of the theory, one can translate these axioms on the chronological products. The Epstein–Glaser approach is an inductive procedure to construct the chronological products in higher orders starting from the first order of the perturbation theory—the interaction Lagrangian—which is a Wick polynomial. For gauge theories one can construct a non-trivial interaction only if one considers a larger Hilbert space generated by the fields associated with the particles of the model and the ghost fields. In this framework the condition of gauge invariance becomes the condition of factorization of the S-matrix to the physical Hilbert space in the adiabatic limit. To avoid infrared problems one works with a formulation of this factorization condition which corresponds to a formal adiabatic limit and it is perfectly rigorously defined [17]. The obstructions to the implementation of the condition of gauge invariance are called anomalies. The most famous is the Adler–Bell–Bardeen–Jackiw anomaly (see [39] for a review). The most convenient way to organize the combinatorial argument seems to be the following one [13, 45]. One constructs the chronological products $T(W_1(x_1), \ldots, W_n(x_n))$ associated with arbitrary Wick monomials $W_1(x_1), \ldots, W_n(x_n)$ according to the Epstein–Glaser prescription [23] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora's prescription [41] (which reduces the renormalization procedure to the process of extension of distributions).

If $T(x)$ is the interaction Lagrangian (i.e. the first order chronological product) and d_Q the BRST operator, we suppose the validity of some 'descent' equations of the type

$$\begin{aligned} &d_Q T(x) = \mathrm{i}\partial_\mu T^\mu(x) && d_Q T^\mu(x) = \mathrm{i}\partial_\nu T^{\mu\nu}(x), \ldots \\ &d_Q T^{\mu_1,\ldots,\mu_{p-1}}(x) = \mathrm{i}\partial_{\mu_p} T^{\mu_1,\ldots,\mu_p}(x) && d_Q T^{\mu_1,\ldots,\mu_p}(x) = 0 \end{aligned} \tag{1.0.1}$$

for some finite p. One denotes by $A^k(x)$, $k = 1, 2, \ldots,$ the expressions $T(x), T^\mu(x), T^{\mu\nu}, \ldots$ and we suppose that these expressions have a well defined ghost number, i.e. all terms of the Wick polynomial $A^k(x)$ have the same ghost degree. Then we can write the preceding equation in the compact form

$$d_Q A^k(x) = \mathrm{i} \sum_m c_m^{k;\mu} \frac{\partial}{\partial x^\mu} A^m(x) \qquad k = 1, 2, \ldots \tag{1.0.2}$$

for some constants $c_m^{k;\mu}$. The gauge invariance condition has the generic form

$$d_Q T(A^{k_1}(x_1), \ldots, A^{k_n}(x_n)) = \mathrm{i} \sum_{l=1}^n (-1)^{s_l} \sum_m c_m^{k_l;\mu} \frac{\partial}{\partial x_l^\mu} T(A^{k_1}(x_1), \ldots, A^m(x_l), \ldots, A^{k_n}(x_n)) \tag{1.0.3}$$

for all $n \in \mathbb{N}$ and all $k_1, \ldots, k_p = 1, 2, \ldots$. Here the expression

$$s_l \equiv \sum_{i=1}^{l-1} gh(A^{k_i}). \tag{1.0.4}$$

One can define [13] the notion of a derivative of a Wick polynomial with respect to the basic fields of the model. In particular one can apply these derivative operators to the polynomials $A^k(x)$, $k = 1, 2, \ldots$.

Then we can prove that the gauge invariance condition can be reduced to some identities verified by the vacuum expectation values of the chronological products of the following type: $\langle \Omega, T(DA^{k_1}(x_1), \ldots, DA^{k_n}(x_n))\Omega\rangle$; here $DA^k(x)$, $k = 1, 2, \ldots,$ are derivatives of the basic expressions $A^k(x)$, $k = 1, 2, \ldots$. These are the so-called C-g identities in the language of [16–19]; it is plausible to expect that they are equivalent to the Ward (Slavnov–Taylor) identities from the usual formulation of gauge theories, so we prefer to call them Ward identities.

This idea was used in [13] to study the conservation of the electromagnetic current in quantum electrodynamics. The generalization of this idea to non-Abelian gauge theories is under investigation [7].

The methods presented in this paper could be a step forward of the proof that the anomalies are absent in higher orders of the perturbation theory. However, if one writes the Ward identity corresponding to say, the axial anomaly in an arbitrary order of the perturbation theory, one immediately sees that if such an anomaly does appear then it cannot be eliminated by some redefinitions of the vacuum averages of chronological products. To prove that such anomalies do not appear it seems that one requires the use of some other new ideas. Such an idea could be the use of a new normalization condition connected with dilation invariance, that is the analogue of the Callan–Symanzik identities. This idea was used to prove the absence of anomalies in higher orders of the perturbation theory assuming the so-called quantum action principle (formalized also in the algebraic renormalization theory) [4] and was applied to the standard model in [38]. The key point of this type of analysis is the appearance of the anomalous dimension. The use of scale invariance in the Epstein–Glaser formalism was investigated in [29]; if a proper definition of the anomalous dimension could be found, the elimination of the anomalies in higher orders of the perturbation could also be solved in the causal formalism.

We start in the next section with a systematic study of the Wick monomials. In particular, we circumvent the complications associated with the signs coming from the fields with Fermi–Dirac statistics using Grassmann variables (following a suggestion in [42]). In section 3 we sketch the framework of the perturbative renormalization theory of Bogoliubov. In section 4 we formulate the notion of general gauge theory and derive the Ward identities. The basic idea is to consider the modulus R of the (free) quantum fields of the model with respect to the ring of the partial derivative operators; a general BRST operator is a graded derivative nilpotent operator $d_Q : R \to R$. Because we work only with free fields the nilpotency is 'on-shell'. Then one replaces (1.0.3) with a more general structure; we suppose that we have a set of Wick polynomials $\mathcal{A}^i(x), i = 1, \ldots, p$, which we organize as a Wick multiplet (a column matrix) $\mathcal{A}$ and some $p \times p$ matrices c^α such that the following relation is true:

$$d_Q \mathcal{A}(x) = \mathrm{i} \sum_\alpha c^\alpha \partial_\alpha \mathcal{A}(x). \tag{1.0.5}$$

Then a definition of gauge invariance for the associated chronological products is possible in a natural way. The usual gauge models are particular cases of this more general structure. In section 5 we check the absence of anomalies in lower orders of the perturbation theory for the Yang–Mills model.

2. The general framework

2.1. Free fields

Here we define the general framework of a free field theory in the Fock space following closely the point of view of [13]. Some standard notions from quantum relativistic mechanics are used [23, 48]: the upper (lower) hyperboloids of mass $m \geqslant 0$ are by definition $X_m^\pm \equiv \{p \in \mathbb{R}^4 |\, \|p\|^2 = m^2 \operatorname{sign}(p_0) = \pm\}$; they are Borel sets with respect to the Lorentz invariant measure $\mathrm{d}\alpha_m^+(p) \equiv \frac{\mathrm{d}\mathbf{p}}{2\omega(\mathbf{p})}$. Here: $\|\cdot\|$ is the Minkowski norm defined by $\|p\|^2 \equiv p \cdot p$, and $p \cdot q$ is the Minkowski bilinear form: $p \cdot q \equiv p_0 q_0 - \mathbf{p} \cdot \mathbf{q}$. We also denote $V^+ \equiv \{x \in \mathbb{R}^4 |\, \|x\|^2 \geqslant 0\}$ and $V^- \equiv \{x \in \mathbb{R}^4 |\, \|x\|^2 \leqslant 0\}$. We define a *system of free fields* to be the ensemble $(\phi^A(x), \mathcal{F}, \Omega, U_{a,L})$ where

(i) $\phi^A(x)$, $A = 1, \ldots, N$, are distribution-valued operators acting in the Fock space $\mathcal{F}$ with a common dense domain D_0. Here $x \in M$ where M is the Minkowski space.

(ii) $\Omega \in D_0$ is called the *vacuum state*. The vectors $\phi^{A_1}(x_1) \cdots \phi^{A_n}(x_n)\Omega$ generate the Fock space $\mathcal{F}$.

(iii) $a, L \mapsto U_{a,L}$ is a unitary representation of the group $SL(2, \mathbb{C})$ acting in $\mathcal{F}$ such that

$$U_{a,L} \phi^A(x) U_{a,L}^{-1} = S(L^{-1})^A{}_B \phi^B(\delta(L) \cdot x + a) \tag{2.1.1}$$

here $SL(2, \mathbb{C}) \ni L \mapsto \delta(L) \in \mathcal{L}_+^\uparrow$ is the covering map and $SL(2, \mathbb{C}) \ni L \mapsto S(L)$ is an $N \times N$ representation of $SL(2, \mathbb{C})$.

(iv) $\mathrm{supp}(\widetilde{\phi^A}) \subset X^+_{M_A} \cup X^-_{M_A}$ where $M_A \geqslant 0$ is called the *mass* of the field ϕ^A, $A = 1, \ldots, N$.

(v) Let us denote by $\phi^A_\pm$ the *positive* (*negative*) *frequency components* of ϕ^A such that $\mathrm{supp}\big(\widetilde{\phi^A_\pm}\big) \subset X^\pm_{M_A}$; then there exists a system of numbers $z_A \in \mathbb{Z}$, $A = 1, \ldots, N$, such that the following *canonical* (*anti*)*commutation relations* are true:

$$\begin{aligned} &\phi^A_\pm(x)\phi^B_\pm(y) = (-1)^{z_A z_B}\phi^B_\pm(y)\phi^A_\pm(x) \\ &\phi^A_\pm(x)\phi^B_\mp(y) - (-1)^{z_A z_B}\phi^B_\mp(y)\phi^A_\pm(x) = D^{AB}_\pm(x-y) \times \mathbf{1}_{\mathcal{F}} \end{aligned} \tag{2.1.2}$$

where $D^{AB}_\pm(x)$ are distributions verifying

$$D^{AB}_\pm(x) = 0 \qquad \text{iff} \quad z_A + z_B \neq 0 \tag{2.1.3}$$

and if we define

$$D^{AB}(x) \equiv D^{AB}_+(x) + D^{AB}_-(x) \tag{2.1.4}$$

then these distributions have causal support: $\mathrm{supp}(D^{AB}(x)) \subset V^+ \cup V^-$.

(vi) One has

$$\phi^A_-(x)\Omega = 0 \qquad \forall A = 1, \ldots, N. \tag{2.1.5}$$

(vii) *Equations of motion* of the type

$$\sum_\alpha u^\alpha_A \partial_\alpha \phi^A(x) = 0 \qquad A = 1, \ldots, N \tag{2.1.6}$$

for some constants u^α_A are verified; here we use Schwartz multi-indices $\alpha, \beta, \ldots$ but one can also use the alternative notation $u^{\mu\nu\cdots}_A$ from jet-bundle extension theory. One cannot avoid the existence of the equations of motion: indeed, because of requirement (iv) the fields will verify the Klein–Gordon equation:

$$\partial^2\phi^A + M_A^2\phi^A = 0 \qquad A = 1, \ldots, N \tag{2.1.7}$$

i.e. the preceding equation is true for

$$u^{\mu\nu}_A = M_A^2 g^{\mu\nu} u_A \tag{2.1.8}$$

for arbitrary numbers u_A. For Dirac fields one has a first order system of equations of motion: the Dirac equation.

(viii) For some of the fields a reality condition might be imposed, connecting the Hermitian conjugates $(\phi^A)^*$ and the original fields ϕ^B.

Let us recall the fact that from (ii) one can derive that if the (graded) commutators of some operator X with all fields ϕ^A are zero, then this operator is proportional to the unit operator $\mathbf{1}$ from the Fock space.

One can easily see that all known models in Fock spaces can be accommodated in this scheme.

We avoid the complications due to the signs from (2.1.2) if we consider a $\mathbb{Z}$-graduated Grassmann algebra $\mathcal{G} = \sum_{n\in\mathbb{Z}} \mathcal{G}_n$ over $\mathbb{C}$ and some Grassmann numbers $g_A \in \mathcal{G}$ which are invertible and of parity z_A, $\forall A = 1, \ldots, N$. Then we consider distributions with values in $\mathcal{G} \otimes \mathcal{L}(\mathcal{F})$ (here $\mathcal{L}(\mathcal{F})$ are the linear operators from $\mathcal{F}$) given by

$$\varphi^A(x) \equiv g_A \otimes \phi^A(x) \qquad \forall A = 1, \ldots, N. \tag{2.1.9}$$

We call these operators the *supersymmetric associated fields*. We consider $J^r(\mathbb{R}^N, M)$ the rth order jet bundle extension of the trivial fibre bundle $\mathbb{R}^N \times M \to M$ which describes classical fields with N components defined over the Minkowski space $M \sim \mathbb{R}^4$ and we consider the jet bundle coordinates u_A^α, $A = 1, \ldots, N$, $|\alpha| \leqslant r$. The natural number r should be chosen large enough. Then we define the following operators:

$$\varphi_u^\pm(x) \equiv \sum_{\alpha,A} u_A^\alpha \partial_\alpha \varphi_\pm^A(x) \tag{2.1.10}$$

and

$$\varphi_u(x) \equiv \varphi_u^+(x) + \varphi_u^-(x). \tag{2.1.11}$$

We call the *mass-shell* the linear subspace:

$$\mathcal{M} \equiv \{u \in J^r(\mathbb{R}^N, M) | \varphi_u = 0\} \tag{2.1.12}$$

and denote $[u] \equiv u$ modulo $\mathcal{M}$. We see that in fact the operators $\varphi_u(x)$ depend only on the equivalence class $[u]$, i.e. we can consistently use the notation:

$$\varphi_{[u]}(x) \equiv \varphi_u(x). \tag{2.1.13}$$

One can easily verify that we have the following form of the canonical commutation relations:

$$\left[\varphi_u^\pm(x), \varphi_w^\pm(y)\right] = 0 \qquad \left[\varphi_u^\pm(x), \varphi_w^\mp(y)\right] = \Delta_{uw}^\pm(x-y) \qquad [\varphi_u(x), \varphi_w(y)] = \Delta_{uw}(x-y) \tag{2.1.14}$$

where on the left-hand side we have the *usual* commutator and we have defined

$$\begin{aligned} \Delta_{uw}^\pm(x-y) &\equiv \sum_{\alpha,A} \sum_{\beta,B} g_A g_B u_A^\alpha w_B^\beta \partial_\alpha^x \partial_\beta^y D_\pm^{AB}(x-y) \\ \Delta_{uw}(x-y) &\equiv \Delta_{uw}^+(x-y) + \Delta_{uw}^-(x-y). \end{aligned} \tag{2.1.15}$$

One can easily see that the distribution $\Delta_{uw}(x-y)$ has causal support and we have the symmetry property

$$\Delta_{uw}(x-y) = \Delta_{wu}(y-x). \tag{2.1.16}$$

It is convenient to choose the Grassmann algebra $\mathcal{G}$ such that $\mathcal{G}_0 = \mathbb{C}$; then we have $\Delta_{uw}^\pm(x-y) \in \mathbb{C}$.

The fact that in (2.1.14) we have the ordinary commutator will make all the following computations more convenient because we will not worry about the Jordan signs appearing for fields with Fermi–Dirac statistics.

In the following we will need a derivative operation defined on the classical field jet bundle: if α is multi-index, then we define $\partial_\alpha : J^r(\mathbb{R}^N, M) \to J^r(\mathbb{R}^N, M)$ according to

$$(\partial_\alpha u)_A^\beta \equiv \sum_{\alpha+\beta=\gamma} u_A^\gamma. \tag{2.1.17}$$

Then we have three elementary facts.

Proposition 2.1. *(i) If $u \in \mathcal{M}$ then $\partial_\alpha u \in \mathcal{M}$ for any multi-index α.*

Proof. One applies the partial derivative operator ∂_α to the equations of motion (2.1.6). □

Proposition 2.2. *The following relations are true:*

$$\partial_\alpha \varphi_u(x) = \varphi_{\partial_\alpha u}(x) \tag{2.1.18}$$

$$\partial_\alpha \Delta_{uw} = \Delta_{\partial_\alpha u, w}. \tag{2.1.19}$$

Proof. The first relation is a result of an elementary computation. For the second relation, we apply the partial derivative operator ∂_α to the last canonical commutation relation (2.1.14) and use the first relation. □

We also note that there is a natural group action $l, u \mapsto l \cdot u$ of the group $SL(2, \mathbb{C})$ on the elements of $J^r(\mathbb{R}^N, M)$. The Hermitian conjugation operation postulated in item (viii) of the preceding subsection induces a natural conjugation operation $u \mapsto u^*$ on the classical fields from $J^r(\mathbb{R}^N, M)$. We will need these operations later.

We end this subsection by pointing out the fact that in the literature one usually uses Grassmann valued classical fields instead of the classical fields from $J^r(\mathbb{R}^N, M)$. The connection is $J^r(\mathbb{R}^N, M) \ni u^\alpha_A \to g_A u^\alpha_A \in J^r(\mathcal{G} \otimes \mathbb{R}^N, M)$.

The classical structure $J^r(\mathbb{R}^N, M)$ associated with the Wick monomial algebra was used somewhat differently in [5] and [13].

2.2. Supersymmetric Wick monomials

A complete and rigorous investigation of the Wick combinatorial arguments can be found in [15]. Here we give an approach which does not use Feynman graphs. We use consistently Bourbaki conventions $\sum_\emptyset \equiv 0, \prod_\emptyset \equiv 1$. We will define Wick monomials through the following proposition.

Proposition 2.3. *The operator-valued distributions $N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big)$ are uniquely determined through the following properties:*

$$N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big)\Omega = \varphi^+_{u_1}(x_1), \ldots, \varphi^+_{u_n}(x_n)\Omega \tag{2.2.1}$$

$$\big[N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big), \varphi_w(y)\big] = \sum_{l=1}^{n} N\big(\varphi_{u_1}(x_1), \ldots, \widehat{\varphi_{u_l}}, \ldots, \varphi_{u_n}(x_n)\big)\Delta_{u_l w}(x_l - y) \tag{2.2.2}$$

$$N(\emptyset) \equiv \mathbf{1}. \tag{2.2.3}$$

In the first two relations n is arbitrary.

Proof. It is elementary. For $n = 1$ we find from the second property that $N(\varphi_u(x)) - \varphi_u(x)$ commutes with every operator $\varphi_w(y)$, so it must be of the form const $\times$ **1**. But the first relation fixes this constant as 0. Next, we suppose that we have defined the expressions $N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_{n-1}}(x_{n-1})\big)$ and we use the second and the first relations to define the action of $N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big)$ on vectors of the type $\varphi_{w_1}(y_1), \ldots, \varphi_{w_k}(y_k)\Omega$; from (ii) of the previous subsection we know that they generate the whole Fock space. □

We call the operators $N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big)$ *supersymmetric Wick* (or *normal*) *monomials in n variables*.

Let us note that, in fact, the Wick monomial $N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big)$ depends only on the equivalence classes $[u_1], \ldots, [u_n]$. Using induction, one can easily prove that it is completely symmetric in the arguments.

We can easily establish the connection with the usual definition of the Wick monomials.

Proposition 2.4. *The following relation is true:*

$$N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big) = \sum_{I,J \in \mathrm{part}\{1,\ldots,n\}} \prod_{i \in I} \varphi^+_{u_i}(x_i) \prod_{j \in J} \varphi^-_{u_j}(x_j). \tag{2.2.4}$$

Proof. We first note that the order of the factors in the two products is irrelevant because of the commutativity property (2.1.14). The proof consists in denoting the right-hand side of the relation by $N'\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n)\big)$ and proving that the three relations appearing in the preceding proposition are true. Then we use the uniqueness assertion. □

As a immediate corollary we obtain the following.

Corollary 2.5. *The following relations are true:*

$$N\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n),\varphi_w(y)\big) = N\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n)\big)\varphi_w(y) - \sum_{l=1}^{n}\big\langle\Omega,\varphi_{u_l}(x_l)\varphi_w(y)\Omega\big\rangle N\big(\varphi_{u_1}(x_1),\ldots,\widehat{\varphi_{u_l}},\ldots,\varphi_{u_n}(x_n)\big) \tag{2.2.5}$$

$$N\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n),\varphi_w(y)\big) = \varphi_w(y)N\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n)\big) + \sum_{l=1}^{n}\Delta^+_{u_l w}(x_l-y)N\big(\varphi_{u_1}(x_1),\ldots,\widehat{\varphi_{u_l}},\ldots,\varphi_{u_n}(x_n)\big). \tag{2.2.6}$$

Proof. The first relation follows immediately from the preceding proposition. If we combine it with the second property from proposition 2.3 then we obtain the second relation. □

If we apply the preceding results we can also obtain using induction the following corollary.

Corollary 2.6. *The normal products can be expressed as*

$$N\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n)\big) = \sum_k\sum_{i_1<\cdots<i_k} d^+(x_1,\ldots,x_n)\varphi_{u_{i_1}}(x_{i_1}),\ldots,\varphi_{u_{i_k}}(x_{i_k}) \tag{2.2.7}$$

where $d^+(x_1,\ldots,x_n)$ are distributions.

In fact, one can express the distributions from the statement as a sum of distributions d^+_G labelled by Feynman graphs [15]. However, we do not need this result here.

Now, a non-trivial observation is that if we formally 'collapse' all arguments $x_1,\ldots,x_n\mapsto x$ in the expression $N\big(\varphi_{u_1}(x_1),\ldots,\varphi_{u_n}(x_n)\big)$ we obtain well-defined operators.

Proposition 2.7. *The expressions*

$$W_{u_1,\ldots,u_n}(x) \equiv N\big(\varphi_{u_1}(x),\ldots,\varphi_{u_n}(x)\big) \tag{2.2.8}$$

are well defined and they are completely symmetric in the indices $u_1,\ldots,u_n$.

Proof. We collapse the arguments in the relations of the preceding proposition and formally obtain

$$W_{u_1,\ldots,u_n}(x)\Omega = \varphi^+_{u_1}(x),\ldots,\varphi^+_{u_n}(x)\Omega \tag{2.2.9}$$

$$\big[W_{u_1,\ldots,u_n}(x),\varphi_w(y)\big] = \sum_{l=1}^{n} W_{u_1,\ldots,\widehat{u_l},\ldots,u_n}(x)\Delta_{u_l w}(x_l-y) \tag{2.2.10}$$

$$W_\emptyset \equiv \mathbf{1}. \tag{2.2.11}$$

The only non-trivial step is to prove that the right-hand side of the first relation is well defined; this is done in [49]. The rest of the proof is identical. □

If $U \equiv \{u_1, \ldots, u_n\}$ then we can use consistently the notation $W_U(x)$. We call expressions of this type *supersymmetric Wick monomials of rank n in one variable*. Again we note that the dependence on the classical fields $u_1, \ldots, u_n$ is only through the equivalence classes. The action $l, u \mapsto l \cdot u$ of the group $SL(2, \mathbb{C})$ on the jet bundle coordinates extends naturally to the action $l, U \mapsto l \cdot U$ componentwise. The same assertion is valid for the Hermitian conjugation $u \mapsto u^*$ which extends componentwise to $U \mapsto U^*$. We now give an elementary result.

Proposition 2.8. *The following formula is true:*

$$\partial_\alpha W_{u_1,\ldots,u_n}(x) = \sum_{l=1}^{n} W_{u_1,\ldots,\partial_\alpha u_l,\ldots,u_n}(x). \tag{2.2.12}$$

Proof. It follows by induction commuting both sides with an arbitrary field $\varphi_w(y)$. □

By definition, a (*supersymmetric*) *Wick polynomial* is any linear combination (with coefficients from $\mathcal{G}$) of Wick monomials. The set of all (supersymmetric) Wick polynomials in the Fock space $\mathcal{F}$ is denoted by sWick($\mathcal{F}$). The action of $SL(2, \mathbb{C})$ and the Hermitian conjugation extend naturally to the set of (supersymmetric) Wick polynomials. When no ambiguity is possible we abandon the attribute supersymmetric.

A generalization of the collapsing procedure used above is available and essential to the perturbation theory. Namely, we consider the expression $N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_k}(x_k)\big)$ for $k > n$ and group the variables $x_1, \ldots, x_k$ in n subsets; then we collapse the arguments to distinct points inside every subset.

Proposition 2.9. *The expression*

$$N\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\big) \equiv N\left(\prod_{u\in U_1} \varphi_u(x_1), \ldots, \prod_{u\in U_n} \varphi_u(x_n)\right) \tag{2.2.13}$$

is well defined and completely symmetric in the arguments.

Proof. As before, we obtain from the first proposition the following relations:

$$N\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\big)\Omega = \prod_{i=1}^{n} \prod_{u\in U_i} \varphi_u^+(x_i)\Omega \tag{2.2.14}$$

$$\big[N\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\big), \varphi_w(y)\big]$$
$$= \sum_{l=1}^{n} \sum_{u\in U_l} N\big(W_{U_1}(x_1), \ldots, W_{U_l-\{u\}}, \ldots, W_{U_n}(x_n)\big)\Delta_{uw}(x_l - y) \tag{2.2.15}$$

$$N(W(x)) \equiv W(x) \tag{2.2.16}$$

and we can use recursion. □

We have results similar to corollary 2.5.

Corollary 2.10. *The following relations are true:*

$$N\big(W_{U_1}(x_1),\ldots,W_{U_k}(x_k),\varphi_w(y)\big) = N\big(W_{U_1}(x_1),\ldots,W_{U_k}(x_k)\big)\varphi_w(y) - \sum_{l=1}^{k}\sum_{u\in U_l}\langle\Omega,\varphi_u(x_l)\varphi_w(y)\Omega\rangle N\big(W_{U_1}(x_1),\ldots,W_{U_l-\{u\}},\ldots,W_{U_k}(x_k)\big) \tag{2.2.17}$$

$$N\big(W_{U_1}(x_1),\ldots,W_{U_k}(x_k),\varphi_w(y)\big) = \varphi_w(y)N\big(W_{U_1}(x_1),\ldots,W_{U_k}(x_k)\big) + \sum_{l=1}^{k}\sum_{u\in U_l}\Delta^{+}_{uw}(x_l-y)N\big(W_{U_1}(x_1),\ldots,W_{U_l-\{u\}},\ldots,W_{U_k}(x_k)\big) \tag{2.2.18}$$

$$\begin{aligned}&\big\langle\Omega, N\big(W_{U_1}(x_1),\ldots,W_{U_k}(x_k)\big)N\big(W_{U_{k+1}}(x_{k+1}),\ldots,W_{U_n}(x_n),\varphi_w(y)\big)\Omega\big\rangle\\ &\quad=\sum_{l=1}^{k}\sum_{u\in U_l}\langle\Omega,\varphi_u(x_l)\varphi_w(y)\Omega\rangle\big\langle\Omega, N\big(W_{U_1}(x_1),\ldots,W_{U_l-\{u\}}(x_l),\ldots,W_{U_k}(x_k)\big)\\ &\qquad\times N\big(W_{U_{k+1}}(x_{k+1}),\ldots,W_{U_n}(x_n)\big)\Omega\big\rangle.\end{aligned} \tag{2.2.19}$$

Proof. We take in corollary 2.5 $\{u_1,\ldots,u_n\} = \cup_{i=1}^{k}U_i$ and we 'collapse' the variables x_j pertaining to the same set U_i. In this way the first two relations follow. The last relation follows from the first one. □

Relation (2.2.15) from the proof of proposition 2.2 is remarkable and deserves a special name. We call an ensemble of operator-valued distributions $E\big(W_{U_1}(x_1),\ldots,W_{U_n}(x_n)\big)$ *of (supersymmetric) Wick type* if and only if the following two conditions are verified:

$$\begin{aligned}&E(\emptyset,\ldots,\emptyset) = \text{const},\\ &E(\emptyset,\ldots,\varphi_u(x_l),\ldots,\emptyset) = \langle\Omega, E(\emptyset,\ldots,\varphi_u(x_l),\ldots,\emptyset)\Omega\rangle\mathbf{1} + E(\emptyset,\ldots,\emptyset)\varphi_u(x_l)\end{aligned} \tag{2.2.20}$$

$$\big[E\big(W_{U_1}(x_1),\ldots,W_{U_n}(x_n)\big),\varphi_w(y)\big] = \sum_{l=1}^{n}\sum_{u\in U_l}E\big(W_{U_1}(x_1),\ldots,W_{U_l-\{u\}},\ldots,W_{U_n}(x_n)\big)\Delta_{uw}(x_l-y). \tag{2.2.21}$$

It is easy to note that if $E\big(W_{U_1}(x_1),\ldots,W_{U_n}(x_n)\big)$ and $F\big(W_{U_{n+1}}(x_{n+1}),\ldots,W_{U_{n+m}}(x_{n+m})\big)$ are expressions of Wick type, then $E\big(W_{U_1}(x_1),\ldots,W_{U_n}(x_n)\big)F\big(W_{U_{n+1}}(x_{n+1}),\ldots,W_{U_{n+m}}(x_{n+m})\big)$ is also an expression of Wick type. The assertion remains true for more than two factors.

We can extend, by linearity, an expression of Wick type to Wick polynomials: if $W_j(x_j)$ are Wick monomials, and $c_{i_1j_1}, c_{i_nj_n} \in \mathcal{G}$ then we define

$$E\left(\sum c_{i_1j_1}W_{j_1}(x_1),\ldots,\sum c_{i_nj_n}W_{j_n}(x_n)\right) \equiv \sum c_{i_1j_1}\cdots c_{i_nj_n}E\big(W_{j_1}(x_1),\ldots,W_{j_n}(x_n)\big) \tag{2.2.22}$$

where the convention about the order of factors is important because of the non-commutativity of the elements of the Grassmann algebra.

The well-known 0-theorem of Epstein–Glaser asserts that expressions of the type

$$E(W_1(x_1),\ldots,W_n(x_n)) \equiv d(x_1,\ldots,x_n)N(W_1(x_1),\ldots,W_n(x_n)) \tag{2.2.23}$$

where $d(x_1, \ldots, x_n)$ is a translation-invariant distribution, are well defined. They are obviously expressions of Wick type. If we take the distribution d of the form

$$d(x_1, \ldots, x_n) = p(\partial)\delta^{n-1}(X) \tag{2.2.24}$$

where $p(\partial)$ is a polynomial in the partial derivatives and

$$\delta^{n-1}(X) \equiv \delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n) \tag{2.2.25}$$

then we obtain some special expressions of Wick type called *quasi-local operators* [8]. Such a type of operator has a distinguished role in the perturbative renormalization theory.

This analysis culminates with an extremely neat form of Wick theorem.

Theorem 2.11. *Let $E\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\big)$ be an expression of Wick type. Then the following relation is valid:*

$$E\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\big) = \sum_{U_i' \subset U_i} \big\langle \Omega, E\big(W_{CU_1'}(x_1), \ldots, W_{CU_n'}(x_n)\big)\Omega\big\rangle N\big(W_{U_1'}(x_1), \ldots, W_{U_n'}(x_n)\big) \tag{2.2.26}$$

where $CU_i' \equiv U_i - U_i'$ are the set-theoretically complements.

Proof. It is shown by induction over the rank $r \equiv |U_1| + \cdots + |U_n|$. For $r = 1$ the formula from the statement is trivial. We suppose that the formula is true for $|U_1| + \cdots + |U_n| = r - 1$ and prove it for $|U_1| + \cdots + |U_n| = r$. One commutes both sides of the identity to be proved with an arbitrary $\varphi_w(y)$ and, using the induction hypothesis, obtains equality. It follows that the relation to be proved is valid up to a constant operator. If we average on the vacuum we obtain that the constant is, in fact, zero. □

We end this subsection by remarking that the set $\mathrm{sWick}(\mathcal{F})$ of all Wick polynomials has a natural Hopf algebra structure:

- the multiplication $m : \mathrm{sWick}(\mathcal{F}) \otimes \mathrm{sWick}(\mathcal{F}) \to \mathrm{sWick}(\mathcal{F})$ is

$$m\big(W_{U_1}, W_{U_2}\big) \equiv W_{U_1 \cup U_2} \tag{2.2.27}$$

- the co-multiplication $\Delta : \mathrm{sWick}(\mathcal{F}) \to \mathrm{sWick}(\mathcal{F}) \otimes \mathrm{sWick}(\mathcal{F})$ is

$$\Delta(W_U) \equiv \sum_{U_1, U_2 \in \mathrm{part}(U)} W_{U_1} \otimes W_{U_2} \tag{2.2.28}$$

- the co-unit $\epsilon : \mathrm{sWick}(\mathcal{F}) \to \mathbb{C}$ is

$$\epsilon(W_U) = \begin{cases} \mathbf{1} & U = \emptyset \\ 0 & U \neq \emptyset \end{cases} \tag{2.2.29}$$

- the antipode operator $S : \mathrm{sWick}(\mathcal{F}) \to \mathrm{sWick}(\mathcal{F})$ is

$$S(W_U) \equiv (-1)^{|U|} W_U. \tag{2.2.30}$$

Then $\mathrm{sWick}(\mathcal{F})$ is a Hopf algebra commutative and co-commutative. This algebra is isomorphic to a Hopf algebra of the type $\mathcal{S}(V)$ with $V = J^r(M, \mathbb{R}^n)$; the isomorphism is

$$\mathcal{S}(V) \ni u_1 \vee \cdots \vee u_n \mapsto W_{u_1, \ldots, u_n} \in \mathrm{sWick}(\mathcal{F}). \tag{2.2.31}$$

For other details see [6].

2.3. Derivatives of Wick polynomials

We can give alternative expressions for this theorem if we introduce the notion of a derivative of a Wick monomial [13]. We give here a more compact treatment. Let us denote the coordinates on the dual of the classical field bundle $(J^r(\mathbb{R}^N, M))^*$ by v^A_α; the duality form is

$$\langle v, u\rangle \equiv \sum_{A,\alpha} v^A_\alpha u^\alpha_A. \tag{2.3.1}$$

We consider the polar of the mass-shell:

$$\mathcal{M}^0 \equiv \{v \in (J^r(\mathbb{R}^N, M))^* | \langle v, u\rangle = 0,\ \forall u \in \mathcal{M}\}. \tag{2.3.2}$$

Then we have the following elementary result.

Proposition 2.12. *Let $v \in \mathcal{M}^0$; the operator $\rho(v) : \mathrm{sWick}(\mathcal{F}) \to \mathrm{sWick}(\mathcal{F})$ is well defined by*

$$\rho(v) W_U(x) \equiv \sum_{u\in U} \langle v, u\rangle W_{U-\{u\}}(x) \tag{2.3.3}$$

and linearity. Moreover, these operators commute among themselves:

$$[\rho(v_1), \rho(v_2)] = 0 \qquad \forall v_1, v_2 \in \mathcal{M}^0. \tag{2.3.4}$$

We call $\rho(v)$ *derivative operators* of Wick polynomials. Because of the commutativity it makes sense to define for any set $V = \{v_1, \ldots, v_n\}$ of elements from $\mathcal{M}^0$ derivative operators of higher order through

$$\rho(V) \equiv \prod_{i=1}^{n} \rho(v_i). \tag{2.3.5}$$

We can provide an alternative expression for the normal products and Wick monomials.

Proposition 2.13. *Let us consider $\{v_j\}_{j\in J}$ a basis in $\mathcal{M}^0$ and $\{v^*_j\}_{j\in J}$ a dual basis in a supplement $\mathcal{M}'$ of $\mathcal{M} \subset J^r(\mathbb{R}^N, M)$ such that the completeness relation is valid:*

$$\sum_{j\in J} (v^*_j)^\alpha_A (v_j)^B_\beta = \delta^\alpha_\beta \delta^B_A. \tag{2.3.6}$$

Then the following formulae are valid:

$$N\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big) = \sum_{j_1,\ldots,j_n\in J} \prod_{k=1}^{n} \langle v_{j_k}, u_k\rangle N\big(\varphi_{v^*_{j_1}}(x_1), \ldots, \varphi_{v^*_{j_n}}(x_n)\big) \tag{2.3.7}$$

$$W_{u_1,\ldots,u_n}(x) = \sum_{j_1,\ldots,j_n\in J} \prod_{k=1}^{n} \big\langle v_{j_k}, u_k\big\rangle W_{v^*_{j_1},\ldots,v^*_{j_n}}(x). \tag{2.3.8}$$

Proof. We use a technique familiar by now. Let us denote the right-hand side of the first relation by $N'\big(\varphi_{u_1}(x_1), \ldots, \varphi_{u_n}(x_n)\big)$ and check that the properties in proposition 2.3 are true. One must use the relation

$$\sum_{j\in J} \langle v_j, u\rangle \Delta_{v^*_j w}(x-y) = \Delta_{uw}(x-y) \tag{2.3.9}$$

which is a consequence of the completeness relation. The second relation from the statement follows if we 'collapse' the arguments into the first one. □

Now we can give two alternative formulations of the Wick theorem. First we have the following theorem.

Theorem 2.14. *Every Wick expression $E(W_1(x_1), \ldots, W_n(x_n))$ (here $W_1(x_1), \ldots, W_n(x_n)$ are Wick polynomials) verifies the following relation:*

$$[E(W_1(x_1), \ldots, W_n(x_n)), \varphi_w(y)] = \sum_{l=1}^{n} \sum_{j\in J} \Delta_{v_j^* w}(x_l - y) E(W_1(x_1), \ldots, \rho(v_j) W_l(x_l), \ldots, W_n(x_n)). \tag{2.3.10}$$

In particular we have for every Wick polynomial:

$$[W(x), \varphi_w(y)] = \sum_{j\in J} \Delta_{v_j^* w}(x - y) \rho(v_j) W(x). \tag{2.3.11}$$

Proof. It is sufficient to consider that $W_1(x_1), \ldots, W_n(x_n)$ are Wick monomials. Then we use the defining relation for an expression of Wick type and relation (2.3.9). □

Now we can give another compact form of Wick theorem.

Theorem 2.15. *The following formula is valid:*

$$E(W_1(x_1), \ldots, W_n(x_n)) = \sum_{V_i} \langle \Omega, E(\rho(V_1) W_1(x_1), \ldots, \rho(V_n) W_n(x_n)) \Omega \rangle N\big(W_{V_1^*}(x_1), \ldots, W_{V_n^*}(x_n)\big) \tag{2.3.12}$$

where the sum runs over all sets V_i of elements of the type v_j $(j \in J)$ from $\mathcal{M}^0$.

Proof. As before, it is sufficient to consider that the expressions W_i are Wick monomials. If we use proposition 2.13 we obtain that the right-hand side of the relation from the statement coincides with the right-hand side of the relation from the Wick theorem (2.2.26). □

We can extend the operation of derivation ∂_α to elements of the polar $\mathcal{M}^0$ through duality: we have

Proposition 2.16. *Let us define $\partial_\alpha : (J^r(\mathbb{R}^N, M))^* \to (J^{r-|\alpha|}(\mathbb{R}^N, M))^*$ according to*

$$(\partial_\alpha v)^A_\beta \equiv v^A_{\alpha+\beta}. \tag{2.3.13}$$

Then

(i) If $v \in \mathcal{M}^0$ we also have $\partial_\alpha v \in \mathcal{M}^0$ for any multi-index α.
(ii) The following commutation relation is valid:

$$[\rho(v), \partial_\alpha] = \rho(\partial_\alpha v). \tag{2.3.14}$$

Proof. The first assertion follows from the corresponding property for the derivatives of the elements of $J^r(\mathbb{R}^N, M)$ and the duality relation:

$$\langle \partial_\alpha v, u \rangle = \langle v, \partial_\alpha u \rangle. \tag{2.3.15}$$

This identity, as well as the last relation from the statement, can be proved directly from the definitions by elementary computations. □

It is convenient to introduce some particular derivatives of the type $\rho(v)$. We consider some field φ^A (the index A is fixed) constrained only by the Klein–Gordon equation (2.1.7). Then we define the elements v_A, $v_A^\mu \in \mathcal{M}^0$, by giving only the *non-zero* entries:

$$(v_A)^B = \delta_A^B \qquad (v_A)^B_{\nu\rho} = -\tfrac{1}{4} M_A^2 \delta_A^B g_{\nu\rho} \tag{2.3.16}$$

and respectively

$$\left(v_A^\mu\right)^B_\nu = \delta_A^B \delta_\nu^\mu \qquad \left(v_A^\mu\right)^B_{\nu\rho\sigma} = -\tfrac{1}{4} M_A^2 \delta_A^B \mathcal{S}_{\nu\rho\sigma} \delta_\nu^\mu g_{\rho\sigma}. \tag{2.3.17}$$

Indeed we immediately have

$$\langle v_A, u\rangle = 0 \qquad \left\langle v_A^\mu, u\right\rangle = 0 \qquad \forall u \in \mathcal{M}.$$

Then we denote

$$\frac{\partial}{\partial \varphi^A} \equiv \rho(v_A) \qquad \frac{\partial}{\partial \varphi^A_\mu} \equiv \rho\left(v_A^\mu\right). \tag{2.3.18}$$

These are in fact derivatives with respect to the basic fields and their first order jet extension.

Next we have the following result [13] following directly from the second formula of theorem 2.14. We say that W_U is a Wick monomial of *first order* if all elements $u \in U$ verify $u_A^\alpha = 0, \forall |\alpha| > 1$. A *first order* Wick polynomial is a sum of first order Wick monomials.

Proposition 2.17. *Suppose that W is a Wick polynomial of first order. Then we have*

$$[W(x), \varphi^A(y)] = \frac{\partial}{\partial \varphi^B} W(x) \Delta^{AB}(x-y) + \frac{\partial}{\partial \varphi^B_\mu} W(x) \partial_\mu \Delta^{AB}(x-y). \tag{2.3.19}$$

The proof is elementary. We also define

$$\partial_\sigma \cdot \frac{\partial}{\partial \varphi^A} \equiv \rho(\partial_\sigma \cdot v_A) \qquad \partial_\sigma \cdot \frac{\partial}{\partial \varphi^A_\mu} \equiv \rho\left(\partial_\sigma \cdot v_A^\mu\right). \tag{2.3.20}$$

Then, for any first order Wick polynomial W we have the following formulae:

$$\partial_\sigma \cdot \frac{\partial}{\partial \varphi^A} W(x) = -\frac{1}{4} M_A^2 \frac{\partial}{\partial \varphi^A_\sigma} W(x) \qquad \partial_\sigma \cdot \frac{\partial}{\partial \varphi^A_\mu} W(x) = \delta_\sigma^\mu \frac{\partial}{\partial \varphi^A} W(x). \tag{2.3.21}$$

We finally note that we can define in a natural way the ghost number of the derivatives $\frac{\partial}{\partial \varphi^A}, \frac{\partial}{\partial \varphi^A_\mu}$ to be z_A. If the elements of the set $V = \{v_1, \ldots, v_k\}$ are of defined ghost number, then

$$gh(V) \equiv \sum gh(v_i). \tag{2.3.22}$$

2.4. Wick monomials

In this subsection, we make the connection with the ordinary Wick monomials defined in the original Fock space $\mathcal{F}$. Loosely speaking, if we strip a supersymmetric Wick monomial of its Grassmann factors in a consistent way, we obtain the usual Wick monomials. First we have the following proposition.

Proposition 2.18. *Let σ be the section of the fibre bundle: $J^r(\mathbb{R}^N, M) \to J^r(\mathbb{R}^N, M)/\mathcal{M}$. Then every Wick monomial can be uniquely written in the form*

$$W_{u_1,\ldots,u_n}(x) = \prod_{i=1}^n (\widetilde{u}_i)^{\alpha_i}_{A_i} \qquad W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) = \prod_{i=1}^n (\widetilde{u}_i)^{\alpha_i}_{A_i} g_{A_i} \qquad \mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) \tag{2.4.1}$$

where $\widetilde{u_i} \equiv \sigma([u_i])$, $W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ *are operator-valued distributions with values in* $\mathcal{G} \otimes \mathcal{L}(\mathcal{F})$ *and* $\mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ *are operator-valued distributions with values in* $\mathcal{L}(\mathcal{F})$. *A similar relation can be established for supersymmetric expressions of Wick type.*

Proof. We define recurrently the expressions $W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ through the following properties:

$$W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)\Omega = \prod_{i=1}^{n} \partial_{\alpha_i}\varphi^{A_i}(x)\Omega \tag{2.4.2}$$

$$\left[W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x), \partial_\beta\varphi^B(x)\right] = \sum_{l=1}^{n} W^{A_1,\ldots,\widehat{A_l},\ldots,A_n}_{\alpha_1,\ldots,\widehat{\alpha_l},\ldots,\alpha_n}(x)\partial^x_{\alpha_l}\partial^y_\beta \Delta^{A_l B}(x-y) \tag{2.4.3}$$

$$W^{\emptyset}_{\emptyset}(x) = \mathbf{1}. \tag{2.4.4}$$

One can prove that these relations define uniquely the expressions $W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ using a familiar argument. Then we obtain the first equality from the statement using the uniqueness argument from proposition 2.7.

The expressions $\mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ can be defined quite similarly and we obtain the second equality from the statement. □

One can see that the expressions $W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ are in fact supersymmetric Wick monomials: they can be obtained for some special choice of the classical fields u_i. Moreover, they are completely symmetric in the couples (A, α) and are not linearly independent. In fact, we have 'equation of motion' of the type

$$\sum_{\alpha_l} u^{\alpha_l}_{A_l} W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) = 0 \qquad \forall u \in \mathcal{M}. \tag{2.4.5}$$

A similar assertion is valid for $\mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$; more precisely, we have skew-symmetry in the couples (A, α) and appropriate equations of motion. The expressions $\mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ are called *Wick monomials* and $W_{u_1,\ldots,u_n}(x)$ is the *associated supersymmetric Wick monomial*. A *Wick polynomial* is an operator acting in $\mathcal{F}$ of the following form:

$$\mathcal{L}(x) = \sum \mathcal{C}^{\alpha_1,\ldots,\alpha_n}_{A_1,\ldots,A_n} \mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) \tag{2.4.6}$$

where $\mathcal{C}^{\alpha_1,\ldots,\alpha_n}_{A_1,\ldots,A_n}$ are complex constants with convenient (anti)-symmetry properties. We denote by Wick$(\mathcal{F})$ the set of Wick polynomials in $\mathcal{F}$. If we express the operators $\mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$ in terms of $W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x)$, then we canonically associate with the Wick polynomial $\mathcal{L}(x)$ a supersymmetric Wick polynomial $L(x)$ acting in $\mathcal{G} \otimes \mathcal{L}(\mathcal{F})$.

We can now define some derivative operators. First, we note that the derivation $\rho(v)$ induces a derivation, also denoted by $\rho(v)$ on the space of Wick polynomials. Next, we have the following result:

Proposition 2.19. *Let us define the operators:* ∂^β_B : sWick → sWick *according to*

$$\partial^\beta_B W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) \equiv \sum_{l=1}^{n} \delta^{A_l}_B \delta^\beta_{\alpha_l} W^{A_1,\ldots,\widehat{A_l},\ldots,A_n}_{\alpha_1,\ldots,\widehat{\alpha_l},\ldots,\alpha_n}(x). \tag{2.4.7}$$

Then the following relations are true:

$$\partial^\beta_B W_U(x) = \sum_{u\in U} \tilde{u}^\beta_B W_{U-\{u\}}(x) \qquad \rho(v) = \sum_{A,\alpha} v^A_\alpha \partial^\alpha_A \qquad \forall v \in \mathcal{M}^0. \tag{2.4.8}$$

3. Perturbation theory in the causal approach

We give here the essential ingredients of the perturbation theory using the supersymmetric formalism described in the preceding section.

3.1. Bogoliubov axioms

We use, essentially, the point of view of of Stora and Fredenhagen [5, 13, 45] using the chronological product. An equivalent point of view uses retarded products [46]. By *perturbation theory* in the sense of Bogoliubov, we mean an ensemble of operator-valued distributions $T(W_1(x_1), \ldots, W_n(x_n)) \in \mathcal{G} \otimes \mathcal{L}(\mathcal{F}), n = 1, 2, \ldots$ called *(supersymmetric) chronological products* (here $W_1(x_1), \ldots, W_n(x_n)$ are supersymmetric Wick polynomials) verifying the following set of axioms.

- Symmetry in all arguments $W_1(x_1), \ldots, W_n(x_n)$.
- Poincaré invariance: for all $(a, L) \in inSL(2, \mathbb{C})$, we have

$$U_{a,L} T(W_1(x_1), \ldots, W_n(x_n)) U_{a,L}^{-1} = T(L \cdot W_1(\delta(L) \cdot x_1 + a), \ldots, L \cdot W_n(\delta(L) \cdot x_n + a)). \tag{3.1.1}$$

  Sometimes it is possible to supplement this axiom by corresponding invariance properties with respect to inversions (spatial and temporal) and charge conjugation. For the standard model only the PCT invariance is available. Also, some other global symmetry with respect to some internal symmetry group might be imposed.
- Causality: if $x_i \geqslant x_j, \forall i \leqslant k,\ j \geqslant k+1$, we then have

$$T(W_1(x_1), \ldots, W_n(x_n)) = T(W_1(x_1), \ldots, W_k(x_k)) T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n)). \tag{3.1.2}$$

- Unitarity: we define the *(supersymmetric) anti-chronological products* according to

$$(-1)^n \bar{T}(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1,\ldots,I_r \in \mathrm{part}(\{1,\ldots,n\})} T_{I_1}(X_1) \cdots T_{I_r}(X_r) \tag{3.1.3}$$

  where we have used the notation

$$T_{\{i_1,\ldots,i_k\}}(x_{i_1}, \ldots, x_{i_k}) \equiv T\big(W_{i_1}(x_{i_1}), \ldots, W_{i_k}(x_{i_k})\big). \tag{3.1.4}$$

  Then the unitarity axiom is

$$\bar{T}(W_1(x_1), \ldots, W_n(x_n)) = T(W_1^*(x_1), \ldots, W_n^*(x_n)). \tag{3.1.5}$$

- The 'initial condition'

$$T(W(x)) = W(x). \tag{3.1.6}$$

Remark 3.1. From (3.1.2) one can derive easily that if we have $x_i \sim x_j, \forall i \leqslant k,\ j \geqslant k+1$ then

$$[T(W_1(x_1), \ldots, W_k(x_k)), T(W_{k+1}(x_{k+1}), \ldots, W_n(x_n))] = 0. \tag{3.1.7}$$

3.2. Epstein–Glaser construction

Epstein–Glaser construction provides an explicit solution for Bogoliubov axioms. We sketch briefly the proof.

Theorem 3.2. *There exists a solution of Bogoliubov axioms.*

Proof. Goes by induction. One supposes that the chronological products are constructed up to the order $n-1$ such that all Bogoliubov axioms are verified. We supplement the induction hypothesis with the requirement that the chronological products (up to the order $n-1$) are expressions of Wick type; this means that we have for all $p = 1, \ldots, n-1$:

$$\left[T\left(W_{U_1}(x_1), \ldots, W_{U_p}(x_p)\right), \varphi_w(y)\right] = \sum_{l=1}^{p}\sum_{u\in U_l} T\left(W_{U_1}(x_1), \ldots, W_{U_l-\{u\}}, \ldots, W_{U_p}(x_p)\right)\Delta_{uw}(x_l - y) \tag{3.2.1}$$

so, according to the Wick theorem 2.11, we have the expansion

$$T\left(W_{U_1}(x_1), \ldots, W_{U_p}(x_p)\right) = \sum_{U_i'\subset U_i} \left\langle\Omega, T\left(W_{CU_1'}(x_1), \ldots, W_{CU_p'}(x_p)\right)\Omega\right\rangle \times N\left(W_{U_1'}(x_1), \ldots, W_{U_p'}(x_p)\right). \tag{3.2.2}$$

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated with arbitrary Wick monomials $W_1, \ldots, W_p$; explicitly

$$\omega(\langle\Omega, T(W_1(x_1), \ldots, W_p(x_p))\Omega\rangle) \leqslant \sum_{l=1}^{p}\omega(W_l) - 4(p-1) \qquad p = 1, \ldots, n-1 \tag{3.2.3}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(W)$ we mean the canonical dimension of the Wick monomial W. It is easy to check that the induction hypothesis is true for $n = 1$.

The construction of Epstein–Glaser is based on the commutator $D\left(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\right)$ with causal support. The explicit expression of this commutator is known in terms of the chronological products up to the order $n-1$:

$$D(W_1(x_1), \ldots, W_n(x_n)) \equiv \sum{}' (-1)^{|Y|}[T_I(X), \bar{T}_J(Y)] \tag{3.2.4}$$

where the sum $\sum'$ goes over the partitions $I \cup J = \{1, \ldots, n\}$, $I \cap J = \emptyset$, $J \neq \emptyset$, $x_n \in I$. Moreover, from the explicit formula of the causal commutator it is clear that this expression is also of Wick type (it is a sum of products of expressions of Wick type—according to the induction hypothesis). So, the Wick theorem can be applied and gives an expression of the type

$$D\left(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\right) = \sum_{U_i'\subset U_i} \left\langle\Omega, D\left(W_{CU_1'}(x_1), \ldots, W_{CU_n'}(x_n)\right)\Omega\right\rangle \times N\left(W_{U_1'}(x_1), \ldots, W_{U_n'}(x_n)\right). \tag{3.2.5}$$

One can show that the order of singularity of the numerical distributions

$$d(x_1, \ldots, x_n) \equiv \left\langle\Omega, D\left(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\right)\Omega\right\rangle \tag{3.2.6}$$

verifies a restriction of the type (3.2.3); this is the content of the so-called *power-counting theorem*. Next, one can provide in a standard way a causal splitting of the distribution $d(x_1, \ldots, x_n)$ such that the Poincaré covariance and the order of singularity are preserved. This induces a causal splitting for the operator-valued distribution

$$D\left(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\right) = A\left(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\right) - R\left(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\right) \tag{3.2.7}$$

which can be used to construct the n-order chronological product $T\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n)\big)$. The unitarity can also be fixed quite easily [23]. The induction is finished. □

From the construction it follows that one can define the chronological products such that we have (3.2.1) and (3.2.3) for all $p = 1, 2, \ldots$. The first relation is the normalization condition (**N3**) of [5, 13]. According to the previous section, we also have alternative formulations for the first two of them, namely, we have for all $n \in \mathbb{N}$:

$$[T(W_1(x_1), \ldots, W_n(x_n)), \varphi_w(y)] = \sum_{l=1}^{n} \sum_{j \in J} \Delta_{v_j^* w}(x_l - y) T(W_1(x_1), \ldots, \rho(v_j) W_l(x_l), \ldots, W_n(x_n)) \tag{3.2.8}$$

and

$$T(W_1(x_1), \ldots, W_n(x_n)) = \sum_{V_i \subset \mathcal{M}^0} \langle \Omega, T(\rho(V_1) W_1(x_1), \ldots, \rho(V_n) W_n(x_n)) \Omega \rangle \times N\big(W_{V_1^*}(x_1), \ldots, W_{V_n^*}(x_n)\big). \tag{3.2.9}$$

We still have some freedom on the chronological products which can be used to impose another condition. Let

$$\Delta_{uw} = \Delta_{uw}^{adv} - \Delta_{uw}^{ret} \tag{3.2.10}$$

be a causal splitting of the distribution with causal support Δ_{uw}. By definition the *Feynman propagator* and the *Feynman anti-propagator* are:

$$\Delta_{uw}^{F} \equiv \Delta_{uw}^{adv} - \Delta_{uw}^{-} = \Delta_{uw}^{ret} + \Delta_{uw}^{+} \qquad \Delta^{AF} \equiv \Delta_{uw}^{+} - \Delta_{uw}^{adv} = -\Delta_{uw}^{ret} - \Delta_{uw}^{-}. \tag{3.2.11}$$

Then we have the following result [45].

Theorem 3.3. *Suppose that the chronological products have been chosen such that they are expressions of Wick type. Then they can be chosen such that one also has for all $n \in \mathbb{N}$:*

$$T\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y)\big) = \sum_{U_i' \subset U_i} \big\langle \Omega, T\big(W_{CU_1'}(x_1), \ldots, W_{CU_n'}(x_n)\big)\Omega \big\rangle \times N\big(W_{U_1'}(x_1), \ldots, W_{U_n'}(x_n), \varphi_w(y)\big) + \sum_{l=1}^{n} \sum_{u \in U_l} \Delta_{uw}^{F}(x_l - y) T\big(W_{U_1}(x_1), \ldots, W_{U_l - \{u\}}, \ldots, W_{U_n}(x_n)\big). \tag{3.2.12}$$

Proof. It is also based on induction on the rank $|U_1| + \cdots + |U_n|$ (see also [5]). One can easily see that the relation from the statement is trivial for $r = 1$. We suppose that it is true for $|U_1| + \cdots + |U_n| = r - 1$ and prove that it can also be fixed for $|U_1| + \cdots |U_n| = r$. We use a familiar technique, namely we consider for this case the commutator of both sides of (3.2.9) with an arbitrary $\varphi_{w'}(z)$ and, using the induction hypothesis, we get zero. So, the relation from the statement for $|U_1| + \cdots + |U_n| = r$ can be affected by a constant 'anomaly':

$$c(x_1, \ldots, x_n, y) \equiv \big\langle \Omega, T\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y)\big)\Omega \big\rangle - \sum_{l=1}^{n} \sum_{u \in U_l} \Delta_{uw}^{F}(x_l - y) \times \big\langle \Omega, T\big(W_{U_1}(x_1), \ldots, W_{U_l - \{u\}}, \ldots, W_{U_n}(x_n)\big)\Omega \big\rangle. \tag{3.2.13}$$

Using the causal factorization property of the chronological products, the induction hypothesis and property (3.2.2), one can prove that the support of the distribution

$c(x_1, \ldots, x_n, y)$ is contained in the diagonal set $x_1 = \cdots = x_n = y$. This means that we have the generic form

$$c(x_1, \ldots, x_n, y) = p(\partial)\delta(x_1 - y) \cdots \delta(x_n - y) \tag{3.2.14}$$

with $p(\partial)$ being some polynomials in the partial derivatives. Moreover, this numerical distribution has convenient covariance properties and a limitation on the degree of p comes from the power counting limitations on the right-hand side of (3.2.13). In the end, it follows that we can absorb the anomaly $c(x_1, \ldots, x_n, y)$ into the vacuum sector of the chronological product $T\big(W_{U_1}(x_1), \ldots, W_{U_n}(x_n), \varphi_w(y)\big)$, $|U_1|+\cdots+|U_n| = r$ without affecting the Epstein–Glaser induction construction. □

In [5, 13] the relation appearing in this proposition is called the normalization condition (**N4**).

As in the preceding section, one can define the chronological products acting in the Hilbert space $\mathcal{F}$ by stripping the Grassmann variables.

4. General gauge theories

4.1. The supersymmetric BRST operator

In the general setting of subsection 2.1 we define a *BRST operator* d_Q on the set of polynomials in the fields $\phi^A_\pm$ through the following properties:

- It gives zero on the constant operator:

$$d_Q \mathbf{1} = 0. \tag{4.1.1}$$

- It is linear over $\mathbb{C}$.
- It acts on the basic fields as follows:

$$d_Q \phi^A_\pm(x) = -\mathrm{i} \sum_{|\alpha| \leqslant s} \sum_B (q^\alpha)^A{}_B \partial_\alpha \phi^B_\pm(x) \tag{4.1.2}$$

  where q^α are real $N \times N$ matrices constrained by

$$(q^\alpha)^A{}_B = 0 \qquad \text{iff} \quad z_B - z_A \neq 1 \tag{4.1.3}$$

  here $s \in \mathbb{N}^*$ is called the *rank* of the BRST operator. The usual case is $s = 1$.
- It is a (graded) derivative operator in the sense that for all $\epsilon_1, \ldots, \epsilon_n = \pm$, we have

$$d_Q\big[\phi^{A_1}_{\epsilon_1}(x_1) \cdots \phi^{A_n}_{\epsilon_n}(x_n)\big] = \sum_{l=1}^n \prod_{i<l} (-1)^{z_{A_i}} \phi^{A_1}_{\epsilon_1}(x_1) \cdots d_Q \phi^{A_l}_{\epsilon_l}(x_l) \cdots \phi^{A_n}_{\epsilon_n}(x_n). \tag{4.1.4}$$

- It commutes with the derivative operators:

$$[d_Q, \partial_\beta] = 0. \tag{4.1.5}$$

It is clear that the usual BRST operator appearing in Yang–Mills models is a particular case of this general framework. One can naturally extend the operator d_Q to the set of polynomials in the fields $\varphi^A_\pm$; then the following properties are true:

- It gives zero on the constant operator:

$$d_Q \mathbf{1} = 0. \tag{4.1.6}$$

- It is linear over $\mathcal{G}$.

- It acts on the basic fields as follows:

$$d_Q \varphi^A_{\pm}(x) = -\mathrm{i} \sum_{|\alpha| \leqslant s} \sum_B (Q^\alpha)^A{}_B \partial_\alpha \varphi^B_{\pm}(x) \tag{4.1.7}$$

  where

$$(Q^\alpha)^A{}_B = g_A g_B^{-1} (q^\alpha)^A{}_B \tag{4.1.8}$$

  and $\deg\big((Q^\alpha)^A{}_B\big) = -1$.
- It is a derivative operator:

$$d_Q\big[\varphi^{A_1}_{\epsilon_1}(x_1)\cdots\varphi^{A_n}_{\epsilon_n}(x_n)\big] = -\mathrm{i}\sum_{l=1}^n (Q^\alpha)^{A_l}{}_B \varphi^{A_1}_{\epsilon_1}(x_1)\cdots\partial_\alpha\varphi^B_{\epsilon_l}(x_l)\cdots\varphi^{A_n}_{\epsilon_n}(x_n). \tag{4.1.9}$$

- It commutes with the derivative operators:

$$[d_Q, \partial_\beta] = 0. \tag{4.1.10}$$

We compute the action of the operator on the supersymmetric fields. We have the following proposition.

Proposition 4.1. *The following formula is true:*

$$d_Q \varphi_u(x) = \zeta^{-1} \varphi_{\zeta Q\cdot u}(x) \tag{4.1.11}$$

where $\zeta \in \mathcal{G}_1$ is a fixed invertible element and we have defined

$$(Q\cdot u)^\alpha_A \equiv -\sum_{\beta+\gamma=\alpha}\sum_B (Q^\beta)^B{}_A u^\gamma_B. \tag{4.1.12}$$

The proof is elementary. We have introduced the factor ζ because $(Q\cdot u)^\alpha_A \in \mathcal{G}_{-1}$ and in this way $\zeta Q\cdot u$ has real values, and it can be considered as an element of the classical manifold $J^r(\mathbb{R}^N, M)$.

If we apply the operator d_Q to the last commutation relation (2.1.14), we obtain

$$\Delta_{\zeta Q\cdot u, w} = -\Delta_{u, \zeta Q\cdot w}. \tag{4.1.13}$$

Another consequence of the preceding proposition is the following corollary.

Corollary 4.2. *The following formulae are true:*

$$d_Q\big[\varphi^{\epsilon_1}_{u_1}(x_1)\cdots\varphi^{\epsilon_n}_{u_n}(x_n)\big] = \mathrm{i}\zeta^{-1}\sum_{l=1}^n \varphi^{\epsilon_1}_{u_1}(x_1)\cdots\varphi^{\epsilon_l}_{\zeta Q\cdot u_l}(x_l)\cdots\varphi^{\epsilon_n}_{u_n}(x_n) \tag{4.1.14}$$

$$d_Q N\big(\varphi^{\epsilon_1}_{u_1}(x_1)\cdots\varphi^{\epsilon_n}_{u_n}(x_n)\big) = \mathrm{i}\zeta^{-1}\sum_{l=1}^n N\big(\varphi^{\epsilon_1}_{u_1}(x_1)\cdots\varphi^{\epsilon_l}_{\zeta Q\cdot u_l}(x_l)\cdots\varphi^{\epsilon_n}_{u_n}(x_n)\big) \tag{4.1.15}$$

$$d_Q W_{u_1,\ldots,u_n}(x) = \mathrm{i}\zeta^{-1}\sum_{l=1}^n W_{u_1,\ldots,\zeta Q\cdot u_l,\ldots,u_n}(x). \tag{4.1.16}$$

$$d_Q W^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) = -\mathrm{i}\sum_{l=1}^n\sum_{B,\beta}(Q^\beta)^{A_l}{}_B W^{A_1,\ldots,A_{l-1},B,A_{l+1},\ldots,A_n}_{\alpha_1,\ldots,\alpha_{l-1},\beta+\alpha_l,\alpha_{l+1},\ldots,\alpha_n}(x) \tag{4.1.17}$$

$$d_Q \mathcal{W}^{A_1,\ldots,A_n}_{\alpha_1,\ldots,\alpha_n}(x) = -\mathrm{i}\sum_{l=1}^n\sum_{B,\beta}\prod_{i<l}(-1)^{z_{A_i}}(q^\beta)^{A_l}{}_B \mathcal{W}^{A_1,\ldots,A_{l-1},B,A_{l+1},\ldots,A_n}_{\alpha_1,\ldots,\alpha_{l-1},\beta+\alpha_l,\alpha_{l+1},\ldots,\alpha_n}(x). \tag{4.1.18}$$

Proof. The first relation follows by direct computations from the derivative property of the operator d_Q, the second relation follows from the first if we use proposition 2.4 and the third one follows if we 'collapse' the variables into the preceding one. The last two relations are direct consequences of the definitions. □

We will have to extend the action of the BRST operator to the dual space of classical fields $(J^r(\mathbb{R}^N, M))^*$. For this we need the following result.

Proposition 4.3.

(i) If $u \in \mathcal{M}$ then $\zeta Q \cdot u \in \mathcal{M}$.
(ii) If $v \in (J^r(\mathbb{R}^N, M))^*$ let us define $Q \cdot v \in (J^{r-s}(\mathbb{R}^N, M))^*$ according to

$$(Q \cdot v)^A_\alpha \equiv \sum_{B,\beta} (Q^\beta)^A{}_B v^B_{\alpha+\beta}. \tag{4.1.19}$$

Suppose now that $v \in \mathcal{M}^0$; then $\zeta Q \cdot v \in \mathcal{M}^0$.
(iii) The following relation is valid:

$$[d_Q, \rho(v)] = \mathrm{i}\zeta^{-1}\rho(\zeta Q \cdot v). \tag{4.1.20}$$

Proof. The proofs of the first two assertions are based on elementary computations. For the last relation, it is sufficient to prove it to be true when applied to a Wick monomial $W_U(x)$. □

4.2. Gauge invariant models

We generalize the framework outlined in the introduction, i.e. we suppose that we have a set of Wick polynomials $\mathcal{A}^i(x)$, $i = 1, \ldots, p$, which we organize as a Wick multiplet (a column matrix) $\mathcal{A}$ and some $p \times p$ matrices c^α such that the following relation is true:

$$d_Q\mathcal{A}(x) = \mathrm{i}\sum_\alpha c^\alpha \partial_\alpha \mathcal{A}(x). \tag{4.2.1}$$

Then we say that we have a *general gauge theory*. If $A^i(x)$ are the supersymmetric Wick polynomials associated with $\mathcal{A}^i(x)$, $i = 1, \ldots, p$, then a similar relation is verified by them:

$$d_Q A(x) = \mathrm{i}\sum_\alpha c^\alpha \partial_\alpha A(x). \tag{4.2.2}$$

We say that a gauge model $A(x)$ is *of degree r* if $c^\alpha = 0, \forall|\alpha| \neq r$. The case considered in the introduction corresponds to $r = 1$.

We have the following consequence.

Proposition 4.4. *Let $A(x)$ be a general gauge theory. Then we also have*

$$\begin{aligned} d_Q[\rho(v_1)\cdots\rho(v_k)A(x)] &= \mathrm{i}\sum_\alpha c^\alpha \partial_\alpha[\rho(v_1)\cdots\rho(v_k)A(x)] \\ &\quad + \sum_{l=1}^k \left[\zeta^{-1}\rho(v_1)\cdots\rho(\zeta Q\cdot v_l)\cdots\rho(v_k)\right. \\ &\quad \left. + \sum_\alpha c^\alpha \rho(v_1)\cdots\rho(\partial_\alpha v)\cdots\rho(v_k)\right] A(x). \end{aligned} \tag{4.2.3}$$

The proof is elementary using the commutation relations (2.3.14) and (4.1.20). One can write the preceding relation more compactly introducing some notation. We denote:

$$Q \cdot \rho(v_1, \ldots, v_k) \equiv \zeta^{-1} \sum_{l=1}^{k} \rho(v_1, \ldots, \zeta Q \cdot v_l, \ldots, v_k) \tag{4.2.4}$$

and

$$\delta \cdot \rho(v_1, \ldots, v_k) \equiv \sum_{l=1}^{k} \sum_{\alpha} c^{\alpha} \rho(v_1, \ldots, \partial_{\alpha} v_l, \ldots, v_k). \tag{4.2.5}$$

We also define

$$D \equiv \sum_{\alpha} c^{\alpha} \partial_{\alpha}. \tag{4.2.6}$$

Then the relation from the preceding proposition can be rewritten as follows:

$$d_Q \rho(V) A(x) = \mathrm{i}(D + Q + \delta) \rho(V) A(x) \tag{4.2.7}$$

where $V = \{v_1, \ldots, v_k\}$.

For further use, we give the following commutation relations:

$$Q \cdot \rho(v, V) - \rho(v) Q \cdot \rho(V) = \zeta^{-1} \rho(\zeta Q \cdot v) \rho(V) \tag{4.2.8}$$

and

$$\delta \cdot \rho(v, V) - \rho(v) \delta \cdot \rho(V) = \sum_{\alpha} c^{\alpha} \rho(\partial_{\alpha} v) \rho(V). \tag{4.2.9}$$

We define a perturbation theory of the general gauge theory $A(x)$. Let us consider a set of chronological products verifying Bogoliubov axioms (and other normalization conditions imposed in the analysis from the preceding section); we say that they verify *gauge invariance of rank k* if the following identity is true for any $|V| = k$:

$$\sum_{V_1, \ldots, V_n \in \mathrm{part}(V)} \Bigg\{ d_Q T(\rho(V_1) A(x_1), \ldots, \rho(V_n) A(x_n)) - \mathrm{i} \sum_{l=1}^{n} [D_l \cdot T(\rho(V_1) A(x_1), \ldots, \rho(V_n) A(x_n)) + T(\rho(V_1) A(x_1), \ldots, (Q + \delta) \cdot \rho(V_l) A(x_l), \ldots, \rho(V_n) A(x_n))] \Bigg\} = 0 \tag{4.2.10}$$

here

$$D_l \equiv \sum_{\alpha} (\mathbf{1} \otimes \cdots \otimes c^{\alpha} \otimes \cdots \mathbf{1}) \partial_{\alpha}^{l}. \tag{4.2.11}$$

It is not very difficult to see that for a Yang–Mills model, the preceding relation for $k = 0$ goes into the usual gauge invariance condition for the chronological product of the Wick monomials $A^i(x)$ so we call this case simply *gauge invariance*. The cases $k > 0$ give the behaviour with respect to the BRST operator of the chronological products of derivatives of the Wick monomials $A^i(x)$. In principle, if one has some non-zero expression on the right-hand side of the preceding relations, one says that there is an *anomaly* of gauge invariance. One should note, however, that by redefining the chronological products one modifies the expression of the anomaly also by terms which are called *co-boundaries* in the spirit of cohomology theory. One speaks of an anomalous gauge theory only if the anomaly is not trivial, i.e. it is not a co-boundary; if it is a co-boundary then it can be eliminated by a redefinition of the chronological product.

There is a connection between gauge invariance of rank k and rank invariance of rank $k+1$ described in the following theorem which is the analogue of the result from appendix B of [13].

Theorem 4.5. *Suppose that $A(x)$ verifies the gauge invariance condition of rank $k+1$. Then the anomalies of the gauge invariance condition of rank k can only appear in the vacuum sector.*

Proof. As in [13] we consider the gauge invariance condition of rank k and commute both sides with an arbitrary $\varphi_w(y)$. Using the induction hypothesis, the Wick theorem and some commutation relations derived before, we get (after some tedious but straightforward computations) the same expressions in both sides. This means that the anomaly is proportional to **1** so it can show up only in the vacuum sector. □

We say that a BRST transformation is *normal* if it can be expressed as the (graded) commutator with some operator Q verifying $Q\Omega = 0$, $Q^*\Omega = 0$. The operator Q is called *supercharge* (or *gauge charge*). In the usual gauge models the BRST transformation is always normal. We now have the following corollary.

Corollary 4.6. *If d_Q is a normal BRST transformation, then the gauge invariance condition is equivalent to the following set of identities:*

$$\sum_{V_1,\ldots,V_n \in \mathrm{part}(V)} \left\{ \sum_{l=1}^{n} D_l \langle \Omega, T(\rho(V_1)A(x_1), \ldots, \rho(V_n)A(x_n))\Omega\rangle \right. \\ \left. + \langle \Omega, T(\rho(V_1)A(x_1), \ldots, (Q+\delta)\cdot\rho(V_l)A(x_l), \ldots, \rho(V_n)A(x_n))\Omega\rangle \right\} = 0 \tag{4.2.12}$$

for any set of derivatives V.

Proof. First we note the fact that gauge invariance of rank k is always true for k large enough. Indeed, the anomaly of the gauge invariance of rank 0 is a quasi-local operator where there is a limitation on the degree of the polynomial in the partial derivatives—see relation (2.2.24); details of the argument can be found, for instance, in [28]. If one considers instead of the Wick polynomials $A^i(x)$ their derivatives $\rho(V_i)A^i(x)$, one can easily see that every derivative lowers the restriction on the degree of the anomaly with at least one unit. This proves the preceding assertion. Now the Ward identities are the vacuum averages of the gauge invariance relations of arbitrary rank. We apply the preceding theorem iteratively and obtain the conclusion. □

It is not so simple to eliminate the anomalies from the vacuum sector. In the case of quantum electrodynamics [13] this can be done using charge conjugation invariance. In the case of a Yang–Mills model, we have some restrictions coming from ghost number counting and PCT invariance, but they do not eliminate all anomalies as we have already proved in another paper [30].

5. The gauge invariance of the Yang–Mills model

In this section we investigate the Ward identities in the lowest orders of the perturbation theory. We will find that they are valid; in fact for order $n \leqslant 3$ this analysis is equivalent to the analysis from [26] and [27].

5.1. Yang–Mills fields

In [25–27] we have considered the following scheme for the standard model (SM): we construct the auxiliary Hilbert space $\mathcal{H}^{gh,r}_{YM}$ from the vacuum Ω by applying the free fields $A_{a\mu}, u_a, \tilde{u}_a, \Phi_a, a = 1, \ldots, r$ and $\psi_A, A = 1, \ldots, N$. The fields ψ_A are in general Dirac fields describing the matter and have the masses $M_A, A = 1, \ldots, N$. We give the spin structure and the statistics for the other fields: first we postulate that $A_{a\mu}$ (respectively $u_a, \tilde{u}_a, \Phi_a, a = 1, \ldots, r$) has vector (respectively scalar) transformation properties with respect to the Poincaré group. In other words, every vector field has three scalar partners. Also $A_{a\mu}, \Phi_a$ are boson and $u_a, \tilde{u}_a, a = 1, \ldots, r$, are fermion fields.

Moreover, if for some index a the vector field A^μ_a has non-zero mass m_a then we suppose that all the other scalar partner fields $u_a, \tilde{u}_a, \Phi_a$ have the same mass m_a.

If for some index a the vector field A^μ_a has zero mass, then the scalar partner fields $u_a, \tilde{u}_a$ also have zero mass but the corresponding scalar field Φ_a can have an arbitrary mass m^*_a or might be absent.

Finally, we admit that for some indices a all the fields $A^\mu_a, u_a, \tilde{u}_a$ might be absent and the corresponding scalar field Φ_a can have an arbitrary mass m^*_a.

The canonical (anti-)commutation relations are

$$\begin{aligned}
&[A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab} g_{\mu\nu} D_{m_a}(x-y) \times \mathbf{1} \qquad && \{u_a(x), \tilde{u}_b(y)\} = \delta_{ab} D_{m_a}(x-y) \times \mathbf{1} \\
&[\Phi_a(x), \Phi_b(y)] = \delta_{ab} D_{m^*_a}(x-y) \times \mathbf{1} && \{\psi_A(x), \overline{\psi_B}(y)\} = \delta_{AB} S_{M_A}(x-y)
\end{aligned} \tag{5.1.1}$$

all other (anti-)commutators are null.

In the Hilbert space $\mathcal{H}^{gh,r}_{YM}$ we suppose given a sesquilinear form $\langle\cdot,\cdot\rangle$ such that

$$A_{a\mu}(x)^\dagger = A_{a\mu}(x) \qquad u_a(x)^\dagger = u_a(x) \qquad \tilde{u}_a(x)^\dagger = -\tilde{u}_a(x) \qquad \Phi_a(x)^\dagger = \Phi_a(x). \tag{5.1.2}$$

The ghost degree is 1 (resp. -1) for the fields u_a (resp. $\tilde{u}_a$), $a = 1, \ldots, r$ and 0 for the other fields.

One can define the BRST *supercharge* Q by

$$\begin{aligned}
&\{Q, u_a\} = 0 \qquad && \{Q, \tilde{u}_a\} = -\mathrm{i}\big(\partial_\mu A^\mu_a + m_a \Phi_a\big) \\
&\big[Q, A^\mu_a\big] = \mathrm{i}\partial^\mu u_a && [Q, \Phi_a] = \mathrm{i} m_a u_a \qquad \forall a = 1, \ldots, r
\end{aligned} \tag{5.1.3}$$

and

$$Q\Omega = 0. \tag{5.1.4}$$

Then one can justify that the *physical* Hilbert space of the Yang–Mills system is a factor space

$$\mathcal{H}^r_{YM} \equiv \mathcal{H} \equiv \mathrm{Ker}(Q)/\mathrm{Ran}(Q). \tag{5.1.5}$$

The sesquilinear form $\langle\cdot,\cdot\rangle$ induces a *bona fide* scalar product on the Hilbert factor space.

Let us consider the set of Wick monomials $\mathcal{W}$ constructed from the free fields $A^\mu_a, u_a, \tilde{u}_a$ and Φ_a for all indices $a = 1, \ldots, r$; we define the BRST operator $d_Q : \mathcal{W} \rightarrow \mathcal{W}$ as the (graded) commutator with the supercharge operator Q. Then one can prove easily that

$$d_Q^2 = 0. \tag{5.1.6}$$

Let us consider the first order Lagrangian (the double dots mean Wick ordering):

$$T(x) \equiv f_{abc}\left[\tfrac{1}{2} : A_{a\mu}(x)A_{b\nu}(x)F_a^{\mu\nu}(x) : - : A_a^{\mu}(x)u_b(x)\partial_\mu\tilde{u}_c(x) :\right] \\ + f'_{abc}\left[:\Phi_a(x)\partial_\mu\Phi_b(x)A_c^{\mu}(x) : -m_b : \Phi_a(x)A_{b\mu}(x)A_c^{\mu}(x) : \right. \\ \left. - m_b : \Phi_a(x)\tilde{u}_b(x)u_c(x):\right] + f''_{abc} : \Phi_a(x)\Phi_b(x)\Phi_c(x) : \\ + j_a^{\mu}(x)A_{a\mu}(x) + j_a(x)\Phi_a(x) \tag{5.1.7}$$

where

$$F_a^{\mu\nu}(x) \equiv \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) \tag{5.1.8}$$

is the Yang–Mills field tensor and the so-called *currents* are

$$j_a^{\mu}(x) =: \overline{\psi_A}(x)(t_a)_{AB}\gamma^\mu\psi_B(x) : + : \overline{\psi_A}(x)(t'_a)_{AB}\gamma^\mu\gamma_5\psi_B(x) : \tag{5.1.9}$$

and

$$j_a(x) =: \overline{\psi_A}(x)(s_a)_{AB}\psi_B(x) : + : \overline{\psi_A}(x)(s'_a)_{AB}\gamma_5\psi_B(x) : \tag{5.1.10}$$

where a number of restrictions must be imposed on the various constants (see [25–27]).

Moreover, if we define

$$T^{\mu}(x) = f_{abc}\left[: u_a(x)A_{b\nu}(x)F_c^{\nu\mu}(x) : -\tfrac{1}{2} : u_a(x)u_b(x)\partial^\mu(x)\tilde{u}_c(x) :\right] \\ + f'_{abc}\left[m_a : A_a^{\mu}(x)\Phi_b(x)u_c(x) : + : \Phi_a(x)\partial^\mu\Phi_b(x)u_c(x):\right] + u_a(x)j_a^{\mu}(x) \tag{5.1.11}$$

and

$$T^{\mu\nu}(x) = \tfrac{1}{2} f_{abc} : u_a(x)u_b(x)F_c^{\nu\mu}(x) : \tag{5.1.12}$$

then we have relation (1.0.1) from the introduction for $p = 2$.

All these Wick polynomials are $SL(2,\mathbb{C})$-covariant, causally commuting and are Hermitian. Moreover, we have the following ghost content:

$$gh(T(x)) = 0 \qquad gh(T^{\mu}(x)) = 1 \qquad gh(T^{\mu\nu}(x)) = 2. \tag{5.1.13}$$

We will construct a perturbation theory verifying Bogoliubov axioms using this set of free fields and imposing the usual axioms of causality, unitarity and relativistic invariance on the chronological products $T(A^{i_1}(x_1), \ldots, A^{i_n}(x_n))$ (where the Wick polynomials $A^i(x)$ must be $T(x)$, $T^{\mu}(x)$ or $T^{\mu\nu}(x)$) such that we have relation (1.0.3) from the introduction which amounts to the factorization property of the chronological products to the physical Hilbert space in the formal adiabatic limit. This generalizes the gauge invariance condition from [1, 21]:

$$d_Q T(T(x_1), \ldots, T(x_n)) = \mathrm{i}\sum_{l=1}^{n} \frac{\partial}{\partial x_l^{\mu}} T\big(T(x_1), \ldots, T_l^{\mu}(x_l), \ldots, T(x_n)\big). \tag{5.1.14}$$

From now on we work with the usual chronological products. The various signs from some of the relations below are obtained by conveniently eliminating the Grassmann variables.

Let us now consider some elements $v_1, \ldots, v_k \in \mathcal{M}^0$ of fixed ghost number and let us define

$$g_l = \sum_{i=1}^{l-1} gh(v_i) \qquad g'_l = \sum_{i=l}^{k} gh(v_i). \tag{5.1.15}$$

Then after some computation, one obtains from (4.2.3):

$$d_Q[\rho(v_1)\cdots\rho(v_k)T(x)] = \mathrm{i}\partial_\mu[\rho(v_1, \ldots, \rho(v_k)T^{\mu}(x)] + \mathrm{i}\sum_{l=1}^{k}[(-1)^{g_l} \\ \times \rho(v_1)\ldots, \rho(q\cdot v_l)\rho(v_k)T(x) + \rho(v_1)\ldots, \rho(\partial_\mu\cdot v_l)\rho(v_k)T^{\mu}(x)] \tag{5.1.16}$$

$$d_Q[\rho(v_1)\cdots\rho(v_k)T^{\mu}(x)] = \mathrm{i}\partial_\nu[\rho(v_1),\ldots,\rho(v_k)T^{\nu\mu}(x)] + \mathrm{i}\sum_{l=1}^{k}[(-1)^{g'_l} \times \rho(v_1)\ldots,\rho(q\cdot v_l)\rho(v_k)T^{\mu}(x) + \rho(v_1)\ldots,\rho(\partial_\nu\cdot v_l)\rho(v_k)T^{\nu\mu}(x)] \tag{5.1.17}$$

$$d_Q[\rho(v_1)\cdots\rho(v_k)T^{\mu\nu}(x)] = \mathrm{i}\sum_{l=1}^{k}(-1)^{g_l}\rho(v_1)\ldots,\rho(q\cdot v_l)\rho(v_k)T^{\mu\mu}(x). \tag{5.1.18}$$

Here the expressions $\partial_\mu\cdot v$ are defined according to (2.3.20) and

$$\begin{array}{lll} q\cdot\dfrac{\partial}{\partial A_{a\mu}} = -\dfrac{1}{4}m_a^2\dfrac{\partial}{\partial\tilde{u}_{a;\mu}} & q\cdot\dfrac{\partial}{\partial u_a} = \dfrac{1}{4}m_a^2 g_{\mu\nu}\dfrac{\partial}{\partial A_{a\mu;\nu}} & q\cdot\dfrac{\partial}{\partial\tilde{u}_a} = 0 \\ q\cdot\dfrac{\partial}{\partial\Phi_a} = m_a\dfrac{\partial}{\partial\tilde{u}_a} & q\cdot\dfrac{\partial}{\partial A_{a\mu;\nu}} = g^{\mu\rho}\dfrac{\partial}{\partial\tilde{u}_a} & q\cdot\dfrac{\partial}{\partial u_{a;\mu}} = -\dfrac{\partial}{\partial A_{a\mu}} \\ q\cdot\dfrac{\partial}{\partial\tilde{u}_{a;\mu}} = 0 & q\cdot\dfrac{\partial}{\partial\Phi_{a;\mu}} = m_a\dfrac{\partial}{\partial\tilde{u}_{a;\mu}} & \end{array} \tag{5.1.19}$$

where the derivatives with respect to the fields are defined according to the general formulae (2.3.18).

Using these relations one can now easily write explicitly all Ward identities. We will not list them here.

5.2. *Lower order Ward identities*

We consider identities (4.2.12) for $2 \leqslant n \leqslant 5$. It follows from the preceding section that we have something non-trivial only if

$$|V| \leqslant 5 \qquad gh(V) = \sum_{l=1}^{n} gh(A^{i_l}) - 1. \tag{5.2.1}$$

Also, in the sum over the partitions of V it is sufficient to consider only those terms for which all subsets $V_1,\ldots,V_n$ are non-void.

The list of these Ward identities is too long to give in detail. We will only mention the choices for the set V and insist on those identities which produce anomalies. Afterwards we will specify the finite renormalizations which do eliminate the anomalies. In all these computations we heavily rely on the various relations verified by the constants appearing in the first order Lagrangian $T(x)$; all these constraints can be found in [26] and [27].

(i) $n = 2$

In the second order perturbation theory we have three possibilities

(i1) $A^{i_1}(x) = A^{i_2}(x) = T(x)$.

In this case we must have $gh(V) = 1$, so we have the following cases:

$$V = \left\{\frac{\partial}{\partial u_a},\frac{\partial}{\partial u_b},\frac{\partial}{\partial\tilde{u}_c},\frac{\partial}{\partial A_{d\rho}}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a},\frac{\partial}{\partial u_b},\frac{\partial}{\partial\tilde{u}_c},\frac{\partial}{\partial\Phi_d}\right\}$$

$$V = \left\{\frac{\partial}{\partial u_a},\frac{\partial}{\partial A_{b\nu}},\frac{\partial}{\partial A_{c\rho}},\frac{\partial}{\partial A_{d\sigma}}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a},\frac{\partial}{\partial A_{b\nu}},\frac{\partial}{\partial A_{c\rho}},\frac{\partial}{\partial\Phi_d}\right\}$$

$$V = \left\{\frac{\partial}{\partial u_a},\frac{\partial}{\partial A_{b\nu}},\frac{\partial}{\partial\Phi_c},\frac{\partial}{\partial\Phi_d}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a},\frac{\partial}{\partial\Phi_b},\frac{\partial}{\partial\Phi_c},\frac{\partial}{\partial\Phi_d}\right\}$$

$$V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d}, \frac{\partial}{\partial \Phi_e}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}\right\} \tag{5.2.2}$$

and the cases obtained from the first six by appending derivatives to one of the fields. In all, there are 24 such relations. We give below only the anomalous Ward identities:

(1) $V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma;\lambda}}\right\}$.

Let us give the Ward identity in detail in the case:

$$\begin{aligned}\frac{\partial}{\partial x_1^\mu}&\left\langle\Omega, T\left(\frac{\partial^2}{\partial u_a \partial A_{b\nu}} T^\mu(x_1), \frac{\partial^2}{\partial A_{c\rho} \partial A_{d\sigma;\lambda}} T(x_2)\right)\Omega\right\rangle \\ &+ \left\langle\Omega, T\left(\frac{\partial^2}{\partial u_a \partial A_{d\sigma}} T^\lambda(x_1), \frac{\partial^2}{\partial A_{b\nu} \partial A_{c\rho}} T(x_2)\right)\Omega\right\rangle \\ &+ (b\nu \leftrightarrow c\rho) + (x_1 \leftrightarrow x_2) + \cdots = 0\end{aligned} \tag{5.2.3}$$

where by $\cdots$ we mean terms which do not produce anomalies.

We will illustrate the procedure of obtaining the anomaly in this case. The first chronological product comes from the causal commutator

$$\left[\frac{\partial^2}{\partial u_a \partial A_{b\nu}} T^\mu(x_1), \frac{\partial^2}{\partial A_{c\rho} \partial A_{d\sigma;\lambda}} T(x_2)\right] = f_{abe} f_{cde} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \partial^\mu D_{m_e}(x_1 - x_2) \tag{5.2.4}$$

and it produces the anomaly

$$f_{abe} f_{cde} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \delta(x_1 - x_2). \tag{5.2.5}$$

The total anomaly produced by the preceding Ward identity is

$$A^{\nu\rho\sigma\lambda}_{1;abcd} = 2\mathrm{i} f_{ade} f_{cbe} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \delta(x_1 - x_2). \tag{5.2.6}$$

(2) $V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}}\right\}$

with the anomaly

$$A^{\nu\rho\sigma}_{2;abcd} = \mathrm{i} f_{abe} f_{cde} (g^{\nu\sigma} \partial^\rho - g^{\nu\rho} \partial^\sigma) \delta(x_1 - x_2) + (b\nu \leftrightarrow c\rho) + (b\nu \leftrightarrow d\sigma) \tag{5.2.7}$$

(3) $V = \left\{\frac{\partial}{\partial u_{a;\nu}}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial A_{c\sigma}}, \frac{\partial}{\partial A_{d\lambda}}\right\}$.

In this case we get an algebraic Ward identity:

$$\begin{aligned}&\left\langle\Omega, T\left(\frac{\partial^2}{\partial A_{a\nu} \partial A_{b\rho}} T(x_1), \frac{\partial^2}{\partial A_{c\sigma} \partial A_{d\lambda\rho}} T(x_2)\right)\Omega\right\rangle \\ &\qquad - \left\langle\Omega, T\left(\frac{\partial^2}{\partial u_a \partial A_{b\rho}} T^\nu(x_1), \frac{\partial^2}{\partial A_{c\sigma} \partial A_{d\lambda\rho}} T(x_2)\right)\Omega\right\rangle \\ &\qquad + (b\rho \leftrightarrow c\sigma) + (b\rho \leftrightarrow d\lambda) + (x_1 \leftrightarrow x_2) + \cdots = 0.\end{aligned} \tag{5.2.8}$$

(4) $V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_{d;\sigma}}\right\}$

with the anomaly

$$A^{\rho\sigma}_{3;abcd} = -2\mathrm{i}g^{\rho\sigma} f'_{dea} f'_{ceb} \delta(x_1 - x_2). \tag{5.2.9}$$

(5) $V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\}$

with the anomaly

$$A^{\rho}_{4;abcd} = -2\mathrm{i}(f'_{dea} f'_{ceb} + f'_{deb} f'_{cea}) \partial^\rho \delta(x_1 - x_2). \tag{5.2.10}$$

(6) $V = \left\{ \frac{\partial}{\partial u_{a;\rho}}, \frac{\partial}{\partial A_{b\sigma}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\}.$

In this case we get an algebraic Ward identity.

(7) $V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\}$

with the anomaly

$$A_{5;abcd} = -12\mathrm{i}\mathcal{S}_{bcd}(f'_{bea} f''_{cde}) \delta(x_1 - x_2) \tag{5.2.11}$$

(i2) $A^{i_1}(x) = T(x)$, $A^{i_2}(x) = T^\nu(x)$

In this case we must have $gh(V) = 2$, so we have the following possibilities:

$$\begin{aligned} &V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \tilde{u}_d} \right\} \qquad && V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\} \\ &V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial \Phi_d} \right\} && V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d} \right\} \\ &V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b} \right\} \end{aligned} \tag{5.2.12}$$

and the cases obtained from the first four by appending a derivative to one of the fields. In all, there are 14 such relations. We give below only the anomalous Ward identities:

(8) $V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma;\lambda}} \right\}$

with the anomaly

$$A^{\nu\rho\sigma\lambda}_{6;abcd} = -\mathrm{i} f_{abe} f_{cde} (g^{\rho\sigma} g^{\nu\lambda} - g^{\rho\lambda} g^{\nu\sigma}) \delta(x_1 - x_2) \tag{5.2.13}$$

(9) $V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial A_{c\rho}}, \frac{\partial}{\partial A_{d\sigma}} \right\}$

with the anomaly

$$A^{\nu\rho\sigma}_{7;abcd} = 2\mathrm{i} f_{abe} f_{cde} (g^{\nu\sigma} \partial^\rho - g^{\nu\rho} \partial^\sigma) \delta(x_1 - x_2) + (b\nu \leftrightarrow c\rho) + (b\nu \leftrightarrow d\sigma) \tag{5.2.14}$$

(10) $V = \left\{ \frac{\partial}{\partial u_{a;\nu}}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial A_{c\sigma}}, \frac{\partial}{\partial A_{d\lambda}} \right\}.$

In this case we get an algebraic Ward identity.

(11) $V = \left\{ \frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial A_{d\rho}} \right\}$

with the anomaly

$$A^{\rho\sigma}_{8;abcd} = -\mathrm{i}g^{\rho\sigma}[m_a(f'_{ead}f'_{ceb} + f'_{ced}f'_{eab}) - (a \leftrightarrow b)]\delta(x_1 - x_2) \tag{5.2.15}$$

(12) $V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_{d;\rho}}\right\}$

with the anomaly

$$A^{\rho}_{9;abcd} = -\mathrm{i}g^{\rho\sigma} f_{abe}f'_{cde}\delta(x_1 - x_2) \tag{5.2.16}$$

(13) $V = \left\{\frac{\partial}{\partial u_{a;\rho}}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d}\right\}$.

In this case we get an algebraic Ward identity.

(i3) $A^{i_1}(x) = T(x)$, $A^{i_2}(x) = T^{\nu\rho}(x)$.

In this case we must have $gh(V) = 3$, so we have the following possibilities:

$$V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial A_{d\rho}}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_d}\right\} \tag{5.2.17}$$

and the cases obtained by appending derivatives to one of the fields. There are six such relations and the corresponding Ward identities do not give anomalies.

(i4) $A^{i_1}(x) = T(x)$, $A^{i_2}(x) = T^{\nu\rho}(x)$.

In this case we also have $gh(V) = 3$, so we have the same possibilities as in case (i3):

$$V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial A_{d\rho}}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_d}\right\} \tag{5.2.18}$$

and the cases obtained by appending derivatives to one of the fields. There are six such relations. The anomalous Ward identities correspond to the following choices:

(14) $V = \left\{\frac{\partial}{\partial u_{a;\sigma}}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial A_{d\sigma}}\right\}$.

In this case we get an algebraic Ward identity

(15) $V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial \Phi_d}\right\}$

with the anomaly

$$A^{\rho\sigma}_{10;abcd} = -\mathrm{i}g^{\rho\sigma}\mathcal{A}_{abc}(f_{abc}f'_{dec})m_e\delta(x_1 - x_2) \tag{5.2.19}$$

(i5) $A^{i_1}(x) = T^{\nu}(x)$, $A^{i_2}(x) = T^{\rho\sigma}(x)$.

In this case we take $gh(V) = 4$, so we have only

$$V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial u_b}, \frac{\partial}{\partial u_c}, \frac{\partial}{\partial u_d}\right\} \tag{5.2.20}$$

and the case obtained by appending derivatives to one of the fields. There are two such relations and the corresponding Ward identities do not produce anomalies.

All anomalies can be removed if we perform the following finite renormalization of the chronological products:

$$T\left(\frac{\partial^2}{\partial u_a \partial A_{d\sigma}}T^{\lambda}(x_1), \frac{\partial^2}{\partial A_{b\nu}\partial A_{c\rho}}T(x_2)\right) \to \cdots + f_{ade}f_{bce}(g^{\rho\sigma}g^{\nu\lambda} - g^{\rho\lambda}g^{\nu\sigma})\delta(x_1 - x_2)$$

$$T\left(\frac{\partial^2}{\partial A_{a\nu}\partial A_{b\rho}}T(x_1), \frac{\partial^2}{\partial A_{c\sigma}\partial A_{d\lambda}}T(x_2)\right) \to \cdots - f_{abe}f_{cde}(g^{\rho\sigma}g^{\nu\lambda} - g^{\rho\lambda}g^{\nu\sigma})\delta(x_1 - x_2)$$

$$
\begin{aligned}
&T\left(\frac{\partial^2}{\partial u_a \partial \Phi_d}T^\sigma(x_1), \frac{\partial^2}{\partial A_{b\rho}\partial \Phi_c}T(x_2)\right) \rightarrow \cdots + f'_{ceb} f'_{dea}(g^{\rho\sigma}g^{\nu\lambda} - g^{\rho\lambda}g^{\nu\sigma})\delta(x_1 - x_2)\\
&T\left(\frac{\partial^2}{\partial A_{a\rho}\partial \Phi_c}T(x_1), \frac{\partial^2}{\partial A_{b\sigma}\partial \Phi_d}T(x_2)\right) \rightarrow \cdots - f'_{ceb} f'_{dea} g^{\rho\sigma}\delta(x_1 - x_2)\\
&T\left(\frac{\partial^2}{\partial \Phi_a \partial \Phi_b}T(x_1), \frac{\partial^2}{\partial \Phi_c\partial \Phi_d}T(x_2)\right) \rightarrow \cdots + \frac{1}{4}\mathcal{S}_{bcd}(f'_{bea} f''_{cde})\delta(x_1 - x_2)\\
&T\left(\frac{\partial^2}{\partial u_a \partial A_{d\sigma}}T^\lambda(x_1), \frac{\partial^2}{\partial u_b\partial A_{c\rho}}T(x_2)\right) \rightarrow \cdots - f_{ace} f_{bde}(g^{\rho\sigma}g^{\nu\lambda} - g^{\rho\lambda}g^{\nu\sigma})\delta(x_1 - x_2)\\
&T\left(\frac{\partial^2}{\partial A_{c\rho} \partial A_{d\sigma}}T(x_1), \frac{\partial^2}{\partial u_a\partial u_b}T^{\mu\nu}(x_2)\right) \rightarrow \cdots + f_{abe} f_{cde}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma})\delta(x_1 - x_2)\\
&T\left(\frac{\partial^2}{\partial \Phi_a \partial u_b}T^\lambda(x_1), \frac{\partial^2}{\partial \Phi_c\partial A_{d\rho}}T^\nu(x_2)\right) \rightarrow \cdots - f'_{ced} f'_{eab} g^{\nu\rho}\delta(x_1 - x_2)\\
&T\left(\frac{\partial^2}{\partial u_a \partial \Phi_d}T^\rho(x_1), \frac{\partial^2}{\partial u_b\partial \Phi_c}T^\nu(x_2)\right) \rightarrow \cdots + f'_{cea} f'_{deb} g^{\rho\sigma}\delta(x_1 - x_2).
\end{aligned}
\tag{5.2.21}
$$

All these renormalizations are made in the vacuum sector, so there is no need to take the vacuum average. Let us note that all these finite renormalizations are consistent with the symmetry properties of the chronological products. If we use formula (3.2.2) we can obtain the finite renormalizations for the original chronological products:

$$
\begin{aligned}
&T(T(x_1), T(x_2)) \rightarrow \cdots + N(x_1)\delta(x_1 - x_2)\\
&T(T^\mu(x_1), T(x_2)) \rightarrow \cdots + N^\mu(x_1)\delta(x_1 - x_2)\\
&T(T^\mu(x_1), T^\nu(x_2)) \rightarrow \cdots + N^{\mu\nu}(x_1)\delta(x_1 - x_2)\\
&T(T(x_1), T^{\mu\nu}(x_2)) \rightarrow \cdots + N^{\mu\nu}(x_1)\delta(x_1 - x_2)
\end{aligned}
\tag{5.2.22}
$$

where

$$
\begin{aligned}
N \equiv\ & \frac{1}{4} f_{abe} f_{cde} : A_{a\mu}A_c^\mu A_{b\rho}A_d^\rho : + \frac{1}{2} f'_{ceb} f_{dea} : A_{a\mu}A_b^\mu \Phi_c \Phi_d :\\
&\qquad + \frac{1}{2m_a} f'_{bea} f''_{cde} : \Phi_a\Phi_b\Phi_c\Phi_d :\\
N^\mu \equiv\ & -f_{ade} f_{bce} : u_a A_b^\mu A_{c\rho} A_d^\rho : - f'_{ebb} f_{eda} : u_a A_b^\mu \Phi_c \Phi_d :\\
N^{\mu\nu} \equiv\ & f_{abe} f_{cde} : u_a u_b A_{c\mu} A_{d\nu} :
\end{aligned}
\tag{5.2.23}
$$

(ii) $n = 3$.

The situation in the third order of the perturbation theory can be analysed as in [27]. One can see that only in two situations can anomalies appear.

(ii1) When the chronological products involve at least one fermionic loop. The relevant choices for the set V are

$$
V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial A_{c\rho}}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\nu}}, \frac{\partial}{\partial \Phi_c}\right\} \qquad V = \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}, \frac{\partial}{\partial \Phi_c}\right\}
\tag{5.2.24}
$$

and other relations with one of the fields differentiated. There are 10 relations of this type. The Ward identities which can produce anomalies correspond only to the choices without derivatives. They are respectively as follows:

for $A^{i_1}(x) = A^{i_2}(x) = A^{i_3}(x) = T(x)$

$$\frac{\partial}{\partial x_1^\mu}\langle\Omega, T\left(j_a^\mu(x_1), j_b^\nu(x_2), j_c^\rho(x_3)\right)\Omega\rangle - m_a\langle\Omega, T\left(j_a(x_1), j_b^\nu(x_2), j_c^\rho(x_3)\right)\Omega\rangle$$
$$+ (b\nu \leftrightarrow c\rho) + (x_1 \leftrightarrow x_2) + (x_1 \leftrightarrow x_3) + \cdots \tag{5.2.25}$$

$$\frac{\partial}{\partial x_1^\mu}\langle\Omega, T\left(j_a^\mu(x_1), j_b^\nu(x_2), j_c(x_3)\right)\Omega\rangle - m_a\langle\Omega, T\left(j_a(x_1), j_b^\nu(x_2), j_c(x_3)\right)\Omega\rangle$$
$$+ (x_1 \leftrightarrow x_2) + (x_1 \leftrightarrow x_3) + \cdots \tag{5.2.26}$$

$$\frac{\partial}{\partial x_1^\mu}\langle\Omega, T\left(j_a^\mu(x_1), j_b(x_2), j_c(x_3)\right)\Omega\rangle - m_a\langle\Omega, T\left(j_a(x_1), j_b(x_2), j_c(x_3)\right)\Omega\rangle$$
$$+ (b \leftrightarrow c) + (x_1 \leftrightarrow x_2) + (x_1 \leftrightarrow x_3) + \cdots \tag{5.2.27}$$

for $A^{i_1}(x) = T^\nu(x),\ A^{i_2}(x) = A^{i_3}(x) = T(x)$

$$\frac{\partial}{\partial x_1^\mu}\langle\Omega, T\left(j_a^\nu(x_1), j_b^\mu(x_2), j_c^\rho(x_3)\right)\Omega\rangle - m_b\langle\Omega, T\left(j_a^\nu(x_1), j_b(x_2), j_c^\rho(x_3)\right)\Omega\rangle$$
$$- (a \leftrightarrow b) + (x_2 \leftrightarrow x_3) + \cdots \tag{5.2.28}$$

for $A^{i_1}(x) = T^\nu(x),\ A^{i_2}(x)T^\rho(x),\ A^{i_3}(x) = T(x)$

$$\mathcal{A}_{abc}\frac{\partial}{\partial x_1^\mu}\langle\Omega, T\left(j_a^\nu(x_1), j_b^\rho(x_2), j_c^\mu(x_3)\right)\Omega\rangle - m_c\langle\Omega, T\left(j_a^\nu(x_1), j_b^\rho(x_2), j_c(x_3)\right)\Omega\rangle$$
$$+ (x_1 \leftrightarrow x_2) + (x_2 \leftrightarrow x_3) + \cdots. \tag{5.2.29}$$

One can show as in [27] that these Ward identities are not anomalous if the axial vertex anomaly vanishes. Indeed, one can show that the preceding Ward identities can be fulfilled if equations (5.1.49)–(5.1.60) from [27] can be causally split; this in turns happens iff the axial anomaly vanishes.

(ii2) We also have some Ward identities where anomalies can appear because of the finite renormalizations (5.2.22). One can easily see that these cases correspond to the choice $A^{i_1}(x) = A^{i_2}(x) = A^{i_3}(x) = T(x)$ and the following assignments for the derivatives V:

$$\begin{aligned} V &= \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\rho}}, \frac{\partial}{\partial A_{c\sigma}}, \frac{\partial}{\partial A_{d\lambda}}, \frac{\partial}{\partial A_{f\nu}}\right\} \\ V &= \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial A_{b\sigma}}, \frac{\partial}{\partial A_{f\lambda}}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d}\right\} \\ V &= \left\{\frac{\partial}{\partial u_a}, \frac{\partial}{\partial \Phi_b}, \frac{\partial}{\partial \Phi_c}, \frac{\partial}{\partial \Phi_d}, \frac{\partial}{\partial \Phi_e}\right\}. \end{aligned} \tag{5.2.30}$$

The first choice gives the Ward identity

$$\frac{\partial}{\partial x_1^\mu}\left\langle\Omega, T\left(\frac{\partial^2}{\partial u_a \partial A_{f\nu}}T^\mu(x_1), \frac{\partial}{\partial A_{b\rho}}T(x_2), \frac{\partial^2}{\partial A_{c\sigma}\partial A_{d\lambda}}T(x_3)\right)\Omega\right\rangle$$
$$+ \mathrm{perm}(b\nu, c\sigma, d\lambda, f\nu) + (x_1, \leftrightarrow x_2) + (x_1, \leftrightarrow x_3) + \cdots = 0. \tag{5.2.31}$$

The chronological product involves the causal splitting of the following commutator

$$\left[\frac{\partial^2}{\partial u_a \partial A_{f\nu}}T^\mu(x_1), T\left(\frac{\partial}{\partial A_{b\rho}}T(x_2), \frac{\partial^2}{\partial A_{c\sigma}\partial A_{d\lambda}}T(x_3)\right)\right]$$
$$= f_{afg} f_{gbd} f_{cde}(g^{\rho\lambda}g^{\nu\sigma} - g^{\rho\sigma}g^{\nu\lambda})\partial^\mu D_{m_g}(x_1 - x_2)\delta(x_2 - x_3) + \cdots \tag{5.2.32}$$

which produces the anomaly

$$A^{\rho\sigma\lambda\nu}_{abcde} = 2 f_{afg} f_{gbd} f_{cde} (g^{\rho\lambda} g^{\nu\sigma} - g^{\rho\sigma} g^{\nu\lambda}) \partial^{\mu} D_{m_g}(x_1 - x_2) + \mathrm{perm}(b\nu, c\sigma, d\lambda, f\nu) = 0. \tag{5.2.33}$$

The other two cases can be treated similarly and do not produce anomalies. Let us note that no finite renormalizations of the third order chronological products are necessary to implement gauge invariance.

(iii) $n = 4, 5$

In these cases, one can argue as in [27] that only when the chronological products involve at least one fermionic loop can one have anomalies. The relevant choices for the derivative set V are similar to the case (ii1) studied above. One obtains that the corresponding Ward identities might be broken by the box and the pentagon anomalies [39].

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